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Abstract

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The purpose of thesis is to present some new results obtained by the author, joint with collaborators or individually, on a collection of nonlinear quasilinear dispersive equations and free boundary problems arising in fluid dynamics. For the former class of problems, we obtain some new low regularity well-posedness results, some of which are optimal in a sense that is going to be discussed in the manuscript, while for the latter, we prove the existence of a class of solutions, known as gallery waves, which preclude the existence of a type of favorable dispersive estimates that are customarily used as a tool in obtaining low regularity well-posedness results.

After giving a brief overview of the main results in Chapter 1, we move on in Chapter 2 to the *irrotational compressible Euler equation in a vacuum setting*:

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0 \text{ in } \Omega \\ \rho(v_t + (v \cdot \nabla)v) + \nabla p = 0 \text{ in } \Omega. \end{cases}$$

Here, the pressure is assumed to be given by constitutive laws of the form $p(\rho) = \rho^{\kappa+1}$. By using the irrotationality condition in the Eulerian formulation of Ifrim and Tataru from [88], we derive a formulation of the problem in terms of the potential function corresponding to the velocity, which turns out to be an acoustic wave equation that also appears in solar seismology. Our object of study is the corresponding linearized problem in a model case, in which our domain is represented by the upper half-space. For this, we investigate the geodesics corresponding to the resulting acoustic metric, which have multiple periodic reflections next to the boundary. Inspired by their dynamics, we define the whispering gallery modes associated to our problem, and prove Strichartz estimates for them. By using a a construction akin to a wave packet, we also prove that one necessarily has a loss of derivatives in the Strichartz estimates for the acoustic wave equation satisfied by the potential function. In particular, this suggests that the low regularity well-posedness result obtained in [88] might

be optimal, at least in a certain frequency regime. To the best of our knowledge, these are the first results of this kind for the irrotational vacuum compressible Euler equations.

In Chapters 3 and 4, which are companion to each other, we consider the well-posedness of the *generalized surface quasi-geostrophic (gSQG) front equation*. Even though the SQG front equation belongs to the same family as the other ones, it has certain special characteristics such as a logarithmic dispersion relation, and for this reason, we treat it separately in Chapter 3, while the rest of the equations from the family are treated in Chapter 4. By using the null structure of the equation via a paradifferential normal form analysis, we obtain balanced energy estimates, which allow us to prove the local well-posedness of the non-periodic gSQG front equation at a low level of regularity (in the SQG case, at only one-half derivatives above scaling), which turns out to be optimal in a sense that is going to be explained in Chapters 3 and 4. In addition, we establish global well-posedness for small and localized rough initial data, as well as modified scattering, by using the testing by wave packet approach of Ifrim-Tataru. Our results improve the ones obtained by Hunter, Shu, and Zhang in [77] and [78], as well as the ones obtained by Gancedo, Gomez-Serrano, and Ionescu in [39].

In Chapter 5, we focus on the *dispersive Hunter-Saxton equation*, which arises in the study of nematic liquid crystals:

$$\begin{cases} u_t + uu_x + u_{xxx} = \dfrac{1}{2}\partial_x^{-1}(u_x^2) \\ u(0) = u_0. \end{cases}$$

Although the equation has formal similarities with the KdV equation, the lack of L^2 control gives it a quasilinear character. Further, the lack of spatial decay obstructs access to dispersive tools, including local smoothing estimates. Here, we give the first proof of local and global well-posedness for the Cauchy problem. Secondly, we improve our well-posedness results with respect to the low regularity of the initial data. The key techniques we use include constructing modified energies to realize a normal form analysis in our quasilinear setting, and frequency envelopes to prove continuous dependence with respect to initial data.

To my family and friends.

Contents

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Secondly, I would like to thank Albert Ai, with whom I have written all of my papers on nonlinear dispersive equations. Throughout our collaboration, he has contributed with key insights, which were crucial to our results. He has also been of tremendous help in elegantly formulating the arguments in our papers.

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This thesis is based on four works which are all either joint works or individual ones, and include the following:

- [14] titled "Counterexamples to Strichartz estimates and gallery waves for the irrotational compressible Euler equation in a vacuum setting". Chapter 2 is almost entirely based on this preprint, which is an individual project.

- [3] titled "Low regularity solutions for the surface quasi-geostrophic front equation". Chapter 3 is almost entirely based on this preprint, which is joint work with Albert Ai.

- [4] titled ”Low regularity well-posedness for the generalized surface quasi-geostrophic front equation”. Chapter 4 is almost entirely based on this preprint, which is joint work with Albert Ai.
- [5] titled ”Well-posedness for the dispersive Hunter-Saxton equation”. Chapter 5 is almost entirely based on this preprint, which is joint work with Albert Ai.

Not present in this thesis is article [6], which is also joint work with Albert Ai.

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Chapter 1

Introduction

One goal of this thesis is to present some new low regularity results for a collection of nonlinear dispersive problems arising in fluid dynamics. Another objective is to prove the existence of a class of solutions to another model, known as gallery waves, which preclude the existence of a type of favorable dispersive estimates that are customarily used as a tool in obtaining low regularity well-posedness results. Some of the aforementioned low regularity well-posedness positive results are optimal in a sense that is going to be elaborated on in Chapters 3 and 4. This text is divided into four main chapters, two of which are companion to each other, while each of the others is concerned with a different model problem. In Chapter 2, we prove the existence of gallery waves for the irrotational compressible vacuum Euler equation in a model case, and in turn show that they force derivative losses in Strichartz estimates; to the best of our knowledge, this is the first result of this kind for the compressible Euler equations. In Chapters 3 and 4, we focus our attention on the equations in the generalized surface quasi-geostrophic front family, and we prove a collection of new low regularity well-posedness, some of which are optimal in a sense that is going to be clarified in the corresponding chapters. In Chapter 5, we turn our attention to the dispersive Hunter-Saxton equation, a model which describes the formation of nematic liquid crystals, for which we also prove a class of low regularity well-posedness results.

The precise meaning of well-posedness can differ in the literature from one problem to another, but in order to give the reader a general idea of which aspects one must analyze, we consider the following simplified initial value problem

$$\begin{aligned} \partial_t u + F(u, \nabla u, \nabla^2 u, \dots, \nabla^k u) = 0, \ \text{in } [0,T] \times \mathbb{R}^d \\ u(0) = u_0, \ \text{in } \mathbb{R}^d \end{aligned} \tag{1.0.1}$$

Here, k is a nonnegative integer, u is assumed to be a scalar or vector-valued function of one time variable and d spatial variables, while F is a smooth nonlinear function. Of course, the domain of u must not necessarily be the whole space $\mathbb{R}^d$, as we shall see in Chapter 2, and F does not necessarily have to be smooth either; however, for the sake of our discussion, we shall keep these assumptions.

One of the first questions that one must consider is whether given an initial data u_0 belonging to a suitable space X_0, the initial value problem (1.0.1) admits a unique solution $u \in C([0,T], X_0)$, where $T > 0$ depends solely on the size of the initial data u_0. For a slew of problems, requiring the solution u to be unique in the space $C([0,T]; X_0)$ turns out to often be a too strong condition. This matter is usually addressed by requiring uniqueness in a subclass X_T that embeds continuously into $C([0,T]; X_0)$; one example of such subclasses consists of the Strichartz spaces, which are going to be discussed in Chapter 2.

Another important question is what happens to the corresponding solutions when we allow the initial data in the space X_0 to vary around a fixed element $u_0 \in X_0$. This is what is known as the question of continuous dependence. To be more specific, what we would like to know is if given a sequence $(u_0)_{n\geq 0}$ with $\lim_{n\to\infty} (u_0)^n = u_0$, is it also going to be true that $\lim_{n\to\infty} u^n = u \in X_T$, where u^n and u are the solutions corresponding to the initial data u_0^n and u_0, respectively. If for a given problem one is able to prove the existence and uniqueness of local solutions, along with their continuity on the initial data, we say that the problem is *locally well-posed in the Hadamard sense.* A closely related question pertains to the nature of the aforementioned continuity. One example of such behavior occurs when we have what we call *L*ipschitz dependence, in the sense that given initial data u_0 and v_0 giving rise to the solutions u and v, one has $\|u - v\|_{X_T} \leq C\|u_0 - v_0\|_{X_0}$. In general, the constant C might depend on u and v. It turns out that this notion is applicable to a significant class of nonlinear problem. The key idea consists of treating the nonlinearity as a perturbative term on a suitable time scale that depends on the size of u and v, and to use Picard iteration or the Contraction Mapping Theorem in order to construct a solution. When this procedure can be carried out, the resulting solutions will also automatically have Lipschitz dependence on the initial data, and we say that the problem is *semilinear.* An example is the nonlinear Schrödinger question, where the power of the nonlinearity is an odd positive integer.

When the solutions described above do not exhibit a Lipschitz dependence on the initial data, but we still have continuous dependence, we call our problem *quasilinear.* In particular, one can not treat the nonlinear terms in the equation perturbatively in the function spaces under consideration, which makes these problems more difficult than the semilinear ones.

All of the equations considered in Chapters 3, 4, and 5 fall into the quasilinear category, and the purpose of the remainder of this introduction is to provide the reader with an expository overview of the main results, while a more technical discussion pertaining to each model will be postponed to their corresponding chapters.

1.1 Free boundary problems

Section 2 is concerned with the contents of [14], which focuses on gallery modes and waves for the irrotational compressible Euler equations in the context of a simplified model. A slew of free boundary problems refer to equations which model fluid or gas dynamics. Some of these models are described by what are known as the compressible Euler equations. In what follows, we are going to be concerned with the evolution of a compressible gas in a physical vacuum setting. In our model, the gas occupies a domain Ω_t with boundary Γ_t at a given time t, and is described by the Eulerian variables (ρ, v), where $\rho \geq 0$ is the density, and v is the velocity. The evolution is described by what are known as the compressible Euler equations

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0 \\ \rho(v_t + (v \cdot \nabla)v) + \nabla p = 0 \end{cases} \tag{1.1.1}$$

In our model, the constitutive laws for the pressure are assumed to have the form

$$p(\rho) = \rho^{\kappa+1}, \kappa > 0. \tag{1.1.2}$$

The equations can be thought of as a coupled system consisting of a wave equation for the variables $(\rho, \nabla \cdot v)$, and a transport equation for the *vorticity* $\omega = \operatorname{curl} v$. Throughout this thesis, we are going to be concerned with the irrotational case, in which $\omega = 0$, and $v = \nabla\phi$, where ϕ is called a *velocity potential*. In turn, this will allow us to reduce our system to a wave equation solved by ϕ. Here, the density ρ is going to be allowed to vanish, a regime known as *vacuum*, so that the gas will occupy the domain $\Omega_t = \{(t, x) | \rho(t, x) > 0\}$, with a moving boundary Γ_t that can be described as

$$\Gamma_t = \{(t, x) | \rho(t, x) = 0\},$$

and towards which ρ decays at 0^+.

In this context, an important quantity is the *speed of sound* c_s, which signifies the speed of propagation of the wave components, and can be defined as

$$c_s^2 = p'(\rho). \tag{1.1.3}$$

Heuristic considerations suggest that there exists a unique stable nontrivial physical regime, which can be described as $d(x, \Gamma_t) \approx c_s^2(t, x)$, which is called *physical vacuum*. It also turns out that the sound speed has a faster decay rate, the particles on the boundary are going to move linearly and independently, which is a scenario that can exist only for a short time. On the other hand, if the speed of sound has a low decay rate, this is consistent with an infinite initial acceleration of the boundary, which is another regime that cannot last for a long time. This means that the physical vacuum regime is the only one that allows the free boundary to move with a bounded velocity and acceleration while interacting with the interior, ensuring that linear waves with speed c_s can reach the free boundary Γ_t in finite time.

Two main approaches have been taken in fluid dynamics: an Eulerian one, in which the reference frame is fixed while the particles are moving, and a Lagrangian one, in which the converse is true, in the sense that the particles are stationary, while the frame is moving. Both of these interpretations have been extensively adopted and analyzed in the context of the compressible Euler equations in the full space $\mathbb{R}^d$, where the local well-posedness problem is well-studied and understood. For a more detailed account of the works in this directions, we refer the reader to Section 2.

Until recently, the free boundary problem corresponding to the physical vacuum had not been studied in the Eulerian setting, and all results had been obtained in Lagrangian coordinates, at high regularity, and in indirectly defined function spaces.

In the Eulerian setting, the first ones to develop a full low regularity theory in the physical vacuum regime were Ifrim and Tataru in [88]. Later, together with Marcelo Disconzi, they proved analogous results for the relativistic Euler equations in a vacuum setting in [47].

In the fluid incompressible case, Ifrim, Pineau, Tataru, and Taylor proved counterparts of the previous results in [83]. We must note that the fluid case poses different difficulties. In particular, one must also assume the Taylor sign condition (this is necessary, as proved by Ebin in [50]), and include the Taylor coefficient in the estimates.

Going back to the compressible case, we define the material derivative D_t as $D_t = \partial_t + v \cdot \nabla$. Physically, this signifies the derivative along a particle flow. The equations (1.1.1) can be rewritten as

$$\begin{cases} D_t\rho + \rho\nabla \cdot v = 0 \\ \rho D_t v + \nabla p = 0 \end{cases} \tag{1.1.4}$$

It turns out that the vorticity $\omega = \operatorname{curl} v$, solves a transport equation given by

$$D_t\omega = -\omega(\nabla v) - (\nabla v)^T\omega.$$

In particular, we deduce that if the vorticity is initially zero, it will remain so, hence the previous equations also model irrotational flows.

By carefully choosing new variables, Ifrim and Tataru were able to recast the equations in the simpler form

$$\begin{cases} D_t r + \kappa r(\nabla \cdot v) = 0 \\ D_t v + \nabla r = 0, \end{cases} \tag{1.1.5}$$

with the linearized counterpart

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa s(\nabla \cdot v) + \kappa r(\nabla \cdot w) = 0 \\ D_t w + w \cdot \nabla v + \nabla s = 0 \end{cases} \tag{1.1.6}$$

We also note that the irrotationality condition will allow us to derive a different formulation for our problem, in terms of a velocity potential ϕ for which $v = \nabla\phi$, and

$$\begin{cases} D_t\phi + r - \dfrac{|v|^2}{2} = 0 \\ D_t r + \kappa r(\nabla \cdot v) = 0 \end{cases} \tag{1.1.7}$$

It turns out that ϕ solves the wave equation

$$D_t^2\phi + \nabla r \cdot \nabla\phi - \kappa r\Delta\phi = 0, \tag{1.1.8}$$

while its linearized counterpart ψ satisfies

$$D_t^2\psi - \nabla r \cdot \nabla\psi - \kappa r\Delta\psi = s(\nabla \cdot v). \tag{1.1.9}$$

We note that we do not impose boundary conditions for the linearized equations. One of the reasons stems from the manner in which the linearized variables (s, w) naturally arise. They are usually defined by considering a one-parameter family of solutions to the system (1.1.5), and then by taking

$$(s, w) = \left(\frac{d}{dh}_{|h=h_0} r^h, \frac{d}{dh}_{|h=h_0} v^h \right),$$

for a fixed value h_0. Now, when h varies, the domain of the solution (r^h, v^h), which is defined as $\{r^h > 0\}$, will also vary, and this precludes us from being able to impose boundary conditions for the linearized system in a meaningful way.

Another reason is that for $\kappa < 1$, when we require the solutions to the previous acoustic wave equation for the linearized velocity potential ψ to belong to a certain energy space, which will be defined in Chapter 2, this condition will already encapsulate significant information, and we would otherwise risk ending up with only trivial solutions; we refer the reader to Chapter 2 for a more detailed discussion of this aspect.

These equations are studied in a scale of weighted Sobolev spaces $\mathscr{H}^{2k}$ whose construction is based on the energy space and powers of the acoustic Laplacian.

In order to understand the scale of Sobolev spaces that Ifrim and Tataru have worked in, we note that our problem admits the scaling law:

$$\begin{aligned} r_\lambda(t, x) &= \lambda^{-2} r(\lambda t, \lambda^2 x) \\ v_\lambda(t, x) &= \lambda^{-1} v(\lambda t, \lambda^2 x), \end{aligned}$$

which points to $\mathscr{H}^{2k_0}$ as the scale invariant space, where $k_0 = \dfrac{d+1}{2} + \dfrac{1}{2\kappa}$.

The main local well-posedness result of Ifrim and Tataru is the following:

Theorem 1.1.1. [Ifrim, Tataru, [88]] The system (1.1.5) is locally well-posed in the space $\mathscr{H}^{2k}$ for $k \in \mathbb{R}$ satisfying

$$k > k_0 + \frac{1}{2}.$$

The wave equation solved by the linearized potential ψ has certain interesting particularities, which stem from the fact that the speed of propagation $c_s^2 = \kappa r$ corresponding to the acoustic metric degenerates towards the free boundary. In order to see them more easily, we consider a simplified model, in which $r = x_d$, $v = 0$, and our domain Ω is the upper half-space in $\mathbb{R}^d$, defined by $\{x \in \mathbb{R}^d | x_d > 0\}$. In this case, the equation satisfied by the linearized potential ψ is

$$\partial_t^2 \psi - \kappa x_d \Delta \psi - \partial_d \psi = 0.$$

We note that this equation also appears in helioseismology, where it describes solar oscillations, and is a simplified scalar equation introduced by Gizon-Barucq-Duruflé-Hanson-Leguèbe-Birch-Chabassier-Fournier-Hohage-Papini in [62]. We refer the reader to Section 2 for a more detailed discussion of the literature.

It turns out that this equation admits time periodic solutions that resemble bump functions, with multiple periodic reflections on the boundary, and which propagate in both the normal and the tangential directions, with small variations in terms of the concentration of the energy. We will call them *gallery modes*, in analogy with the highly localized waves traveling along the boundary in convex domains in the study of the wave and Schrödinger equations. They were studied by Ivanovici in [90] in the context of finding counterexamples for Strichartz estimates in strictly convex domains with boundary; for a more detailed account of the works in this direction, we refer the reader to Section 2. When we are localized in a boundary layer close to the boundary, a gallery wave spends a positive proportion of time in here, in a sense that will be made precise in Section 2. In turn, the solution will fail to exhibit local energy decay, and it will also force derivative losses in the Strichartz estimates.

In what follows, we shall formulate one instance of our main results; for the full and precise statements, the reader is referred to Section 2.

Theorem 1.1.2 (A.,[14])**.** For every $k \in \mathbb{R}$ with $k < \dfrac{d+1}{2}$, there exists a sequence of solutions $(\psi^j)_{j\geq 0}$ to the initial value problem

$$\begin{aligned}(\partial_t^2 - \kappa x_d \Delta - \partial_d)\psi^j &= 0\\ (\psi^j, \partial_t \psi^j)(0) &= (\psi_0^j, \psi_1^j),\end{aligned}$$

with initial data localized in tangential frequency, such that

$$\sup_{j\geq 0}(\|(\psi_1^j, \nabla_x \psi_0^j)\|_{\dot{H}^k \times \dot{H}^{k-1}}) \leq 1,$$

while

$$\lim_{j\to\infty} \|\nabla^2 \psi^j(t,\cdot)\|_{L_t^2 L_x^\infty([0,1]\times\Omega)} = \infty.$$

This suggests that Ifrim and Tataru's result, Theorem 1.1.1 from [88], might be optimal at least in a certain frequency regime.

1.2 Surface quasi-geostrophic front equations

In both Sections 3 and 4, which are companion to each other, we are concerned with the preprints [3] and [4]. The generalized surface quasi-geostrophic (gSQG) equations are a one parameter family of active scalar equations parameterized by a transport term, given by

$$\theta_t + u \cdot \nabla\theta = 0, \qquad u = (-\Delta)^{-1+\frac{\alpha}{2}}\nabla^\perp\theta, \qquad \alpha \in [0,2). \tag{1.2.1}$$

Here, θ represents a scalar evolution on $\mathbb{R}^2$, $(-\Delta)^{-1+\frac{\alpha}{2}}$ is a fractional Laplacian, and $\nabla^\perp$ is given by $\nabla^\perp = (-\partial_y, \partial_x)$.

The case $\alpha = 0$ corresponds to the two-dimensional incompressible Euler equation, while the case $\alpha = 1$ gives the surface quasi-geostrophic equation (SQG) equation. The latter arises as a model for quasi-geostrophic flows confined to a surface in atmospheric and oceanic science. It also shares some similarities with the three dimensional incompressible Euler equation, and thus is often used as a simplified model problem. In particular, the question of finite time singularity formation remains open for both equations. For further analysis of the SQG equation, see Resnick [120].

Front solutions to (1.2.1) are solutions of the form

$$\theta(t,x,y) = \begin{cases} \theta_+ & \text{if } y > \varphi(t,x), \\ \theta_- & \text{if } y < \varphi(t,x), \end{cases}$$

where the front refers to the graph $y = \varphi(t,x)$ with $x \in \mathbb{R}$. Front solutions are closely related to patch solutions, which have the form

$$\theta(t,x,y) = \begin{cases} \theta_+ & \text{if } (x,y) \in \Omega(t), \\ \theta_- & \text{if } (x,y) \notin \Omega(t), \end{cases}$$

where $\Omega(t)$ is a bounded, simply connected domain.

In the generalized $\alpha \neq 1$ case, the equation for the front takes the form

$$\begin{aligned} (\partial_t - c(\alpha)|D_x|^{\alpha-1}\partial_x)\varphi &= Q(\varphi, \partial_x\varphi), \\ \varphi(0,x) &= \varphi_0(x), \end{aligned} \tag{1.2.2}$$

while in the SQG case $\alpha = 1$, the equation takes the form

$$\begin{aligned} (\partial_t - 2\log|D_x|\partial_x)\varphi &= Q(\varphi, \partial_x\varphi), \\ \varphi(0,x) &= \varphi_0(x). \end{aligned} \tag{1.2.3}$$

Here, φ is a real-valued function $\varphi : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $c(\alpha)$ denotes the constant

$$\begin{aligned} c(\alpha) &= -2\sin\left(\frac{\pi(2-\alpha)}{2}\right)\Gamma(1-\alpha), && \alpha \in (0,2), \\ c(\alpha) &= -\frac{1}{2}, && \alpha = 0, \end{aligned} \tag{1.2.4}$$

and the nonlinearity Q is given in the two cases $\alpha \in (0,2)$ and $\alpha = 0$ respectively by

$$Q(f,g)(x) = \int \left(\frac{1}{|y|^\alpha} - \frac{1}{(y^2 + (f(x+y)-f(x))^2)^{\frac{\alpha}{2}}}\right) \cdot (g(x+y) - g(x))\,dy \tag{1.2.5}$$

and

$$Q(f,g)(x) = \int \frac{1}{2\pi}\log\left(1 + \left(\frac{f(x+y)-f(x)}{y}\right)^2\right) \cdot (g(x+y)-g(x))\,dy. \tag{1.2.6}$$

The nonlinearity Q can be written more succinctly using difference quotients,

$$Q(f,g)(x) = \int \frac{1}{|y|^{\alpha-1}} F(\delta^y f) \cdot |\delta|^y g \, dy, \tag{1.2.7}$$

where

$$\delta^y f(x) = \frac{f(x+y)-f(x)}{y}, \qquad |\delta|^y g(x) = \frac{g(x+y)-g(x)}{|y|},$$

and

$$F(s) = 1 - \frac{1}{(1+s^2)^{\frac{\alpha}{2}}} \quad \text{when } \alpha \in (0,2), \qquad F(s) = \frac{1}{2\pi}\log(1+s^2) \quad \text{when } \alpha = 0.$$

The equations (1.2.2) and (1.2.3) are invariant under the scaling

$$t \to \kappa^\alpha t, \qquad x \to \kappa x, \qquad \varphi \to \kappa\varphi,$$

when $\alpha \neq 1$, and

$$t \to \kappa t, \qquad x \to \kappa(x + 2t\log|\kappa|), \qquad \varphi \to \kappa\varphi,$$

when $\alpha = 1$, which implies that $\dot{H}^{\frac{3}{2}}(\mathbb{R})$ is the corresponding critical Sobolev space.

The SQG front equation is a fully nonlinear problem, and our starting point is a paper written by John Hunter, Jingyang Shu, and Qintiang Zhang, in which they prove local well-posedness for data in $H^s_x(\mathbb{R})$ satisfying a smallness condition, where $s \geq 5$. Our first contribution was to find a better paradifferential formulation of the problem; this in turn allowed us to both lower the regularity of the initial data and improve our understanding of its local and global dynamics.

Namely, we proved the following local result:

Theorem 1.2.1 (Ai, A. [3]). Equation (1.2.2) is locally well-posed for initial data in H^s with $s > \frac{5}{2}$. Precisely, for every $R > 0$, there exists $T = T(R) > 0$ such that for any $\varphi_0 \in H^s$ with $\|\varphi_0\|_{H^s} < R$, the Cauchy problem (1.2.3) has a unique solution $\varphi \in C([0,T], H^s)$. Moreover, the solution map $\varphi_0 \mapsto \varphi$ from H^s to $C([0,T], H^s)$ is continuous.

Our main strategy consists of obtaining energy estimates of the form

$$\frac{d}{dt}E^{(s)}(\varphi) \lesssim_A B^2 \cdot E^{(s)}(\varphi),$$

where, morally,

$$A = \|\varphi_x\|_{L^\infty_x}$$
$$B = \|\varphi_x\|_{W^{1,\infty}_x}$$

We also proved that the problem is globally well-posed when the initial data is small and localized in H^s, where $s > 4$, in a sense that is briefly described in Remark 1.2.3.

Subsequently, by observing a null structure satisfied by the SQG and gSQG equations ((1.2.3) and (1.2.2), respectively), we further improved the low regularity thresholds for both the local and global results, while also improving the low frequency threshold to $\dot{H}^{s_0}$ for any $s_0 < \frac{3}{2}$ in the former case, and $s_0 < \min\{1, \alpha\}$ in the latter. In particular, this establishes local well-posedness without requiring control at the level of L^2. These are all substantial improvements over the results in [77, 78, 39, 3].

Our key observation is that both the SQG and gSQG front equations ((1.2.3) and (1.2.2), respectively) exhibit a non-resonance structure when $\alpha \in (0, 2)$. More precisely, we can approximate the nonlinearity Q by

$$Q(\varphi, v) \approx \Omega(\psi, v), \qquad \psi := \partial_x^{-1} F(\varphi_x) \tag{1.2.8}$$

where Ω is a bilinear form whose symbol is given by the resonance function

$$\Omega(\xi_1, \xi_2) = \frac{1}{\alpha}(\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2)),$$

where $\omega(\xi) = c(\alpha) i\xi|\xi|^{\alpha-1}$ when $\alpha \neq 1$, and $\omega(\xi) = 2i\xi \log|\xi|$ when $\alpha = 1$.

This is meaningful because ψ solves an equation similar to φ (see Propositions 3.4, part *b*), in [6, 4]).

We also say that our nonlinearity Q exhibits a *null structure*. This enables us to use normal form methods to obtain *balanced energy estimates*, which use control norms with an even balance of derivatives,

$$\frac{d}{dt} E^{(s)}(\varphi) \lesssim_A B^2 \cdot E^{(s)}(\varphi),$$

where

$$A = \|\partial_x \varphi\|_{L^\infty},$$

and

$$B = \|\partial_x \varphi\|_{C^{\frac{1}{2}+}}$$

for $\alpha = 1$, and

$$B = \|\partial_x \varphi\|_{B^{\frac{\alpha}{2}}_{\infty,2} \cap BMO^{\frac{\alpha}{2}}}$$

for $\alpha \in (0, 1) \cup (1, 2)$.

We can now state our main improved local well-posedness result:

Theorem 1.2.2 (Ai, A. [3, 4])**.** Let $\alpha \in [0, 2)$. Equation (1.2.2) is locally well-posed for initial data in $\dot{H}^{s_0} \cap \dot{H}^s$ with $s_0 < \frac{3}{2}$ and

$$\begin{aligned} s &> \frac{\alpha+3}{2} \qquad \text{if} \quad \alpha = 0, 1, \\ s &\geq \frac{\alpha+3}{2} \qquad \text{if} \quad \alpha \in (0,1) \cup (1,2). \end{aligned}$$

Precisely, for every $R > 0$, there exists $T = T(R) > 0$ such that for any $\varphi_0 \in \dot{H}^{s_0} \cap \dot{H}^s$ with $\|\varphi_0\|_{\dot{H}^{s_0} \cap \dot{H}^s} < R$, the Cauchy problem (1.2.3) has a unique solution $\varphi \in C([0,T], \dot{H}^{s_0} \cap \dot{H}^s)$. Moreover, the solution map $\varphi_0 \mapsto \varphi$ from $\dot{H}^{s_0} \cap \dot{H}^s$ to $C([0,T], \dot{H}^{s_0} \cap \dot{H}^s)$ is continuous.

Remark 1.2.1. In the cases $\alpha \in (0,1) \cup (1,2)$, the control norms A and B allow us to obtain the local well-posedness in the endpoint case $s = \frac{\alpha+3}{2}$. In contrast, in the SQG and 2D Euler cases, we only prove the result in the case $s > 2$ and $s > 3/2$ respectively, due to the logarithmic loss generated by the dispersion relation and nonlinearities.

Remark 1.2.2. We also note that up to the endpoint, this result is sharp as long as energy estimates are concerned. In order to see this, we consider the nonlinearity $\partial_x A_\varphi v$, which appears in the linearized equation, with linearized variables v. We assume that we want to bound the map $v \to \partial_x A_\varphi v$ in L^2, and that for the time being, we can redistribute the arising derivatives in $\partial_x A_\varphi v$ as we wish. We have

$$\partial_x A_\varphi v = \partial_x \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi)|\delta|^y v \, dy,$$

and we can see that, morally speaking, we have $\alpha + 2$ derivatives, that we want to distribute onto the two φ factors (we recall that F is quadratic). In order to achieve this by requiring as little regularity from φ as possible, it is clear that the $\alpha + 2$ derivatives have to be distributed equally between the two factors. This means that we would need the norm $\|"|D_x|^{1+\frac{\alpha}{2}}"\varphi\|_{L^\infty_x}$ to be finite, which imposes the condition $s \geq \dfrac{\alpha+3}{2}$.

Normal forms were introduced by Shatah [123], who used them to prove results on the long-time dynamics of solutions to dispersive equations. Unfortunately, this method cannot be applied directly to quasilinear problems, because the resulting transformations would be unbounded. To address this issue, several approaches have been introduced; we rely on two in the current paper. The first consists of carrying out the analysis in a paradifferential manner, and was introduced by Alazard-Delort [12] in a paradiagonalization argument to obtain Sobolev estimates for the solutions of the water waves equations in the Zakharov formulation. Paradifferential normal forms, coupled with an exponential Jacobian conjugation, were also subsequently employed by Ifrim-Tataru [85] to obtain a new proof of L^2 global well-posedness for the Benjamin-Ono equation, a result first obtained in [89].

The second method consists of employing modified energies instead of applying the direct normal form at the level of the equation. This procedure was introduced by Hunter-Ifrim-Tataru-Wong [71] to study the long time behavior of solutions to the Burgers-Hilbert equation.

Balanced energy estimates were first introduced by Ai-Ifrim-Tataru in [7] in order to study the low regularity local well-posedness of the gravity water waves system. They then later used this type of estimate together with Strichartz estimates to obtain low regularity well-posedness results for the time-like minimal surface problem in the Minkowski space [8].

We next considered global well-posedness for small and localized data. To describe localized solutions, we define the operators

$$L = x + t\alpha c(\alpha)|D_x|^{\alpha-1},$$

when $\alpha \neq 1$, and

$$L = x + 2t + 2t\log|D_x|,$$

when $\alpha = 1$.

They both commute with the associated linear flows, and at time $t = 0$, they both amount to multiplication by x. Then we define the time-dependent weighted energy space

$$\|\varphi\|_X := \|\varphi\|_{\dot{H}^{s_0}\cap\dot{H}^s} + \|L\partial_x\varphi\|_{L^2},$$

where $s > \alpha + 2$ and $s_0 < \min\{1, \alpha\}$. To track the dispersive decay of solutions, we define the pointwise control norm

$$\|\varphi\|_Y := \||D_x|^{1-\delta}\langle D_x\rangle^{\frac{\alpha}{2}+2\delta}\varphi\|_{L^\infty_x}.$$

Theorem 1.2.3 (Ai, A.,[3, 4])**.** Consider data φ_0 with

$$\|\varphi_0\|_X \lesssim \epsilon \ll 1.$$

Then the solution φ to (1.2.2) for $\alpha > 0$ with initial data φ_0 exists globally in time, with energy bounds

$$\|\varphi(t)\|_X \lesssim \epsilon t^{C\epsilon^2}$$

and pointwise bounds

$$\|\varphi(t)\|_Y \lesssim \epsilon\langle t\rangle^{-\frac{1}{2}}.$$

Furthermore, the solution φ exhibits a modified scattering behavior, in a sense whose precise explanation will be omitted in order to avoid technicalities. However, in a nutshell, the key property is that the resulting solutions, at fixed frequency and velocity, tend to behave like solutions to the associated linear flow on the long run.

Remark 1.2.3. For comparison, the main global well-posedness result that we proved in [6] pertains to the case $\alpha = 1$, subject to the conditions $s_0 = 0$, and $s > 4$. We also used a different pointwise control norm there:

$$\|\varphi\|_Y := \||D_x|^{3/4-\delta}\varphi\|_{L^\infty_x} + \||D_x|^{2+\delta}\varphi\|_{L^\infty_x}$$

1.3 The dispersive Hunter-Saxton equations

In Section 5, we are going to be concerned with the contents of the article [5]. We consider the Cauchy problem for the dispersive Hunter-Saxton equation

$$\begin{cases} u_t + uu_x + u_{xxx} = \dfrac{1}{2}\partial_x^{-1}(u_x^2), \\ u(0) = u_0, \end{cases} \tag{1.3.1}$$

where u is a real-valued function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$. Due to the Galilean invariance of (1.3.1), we may fix a definition for ∂_x^{-1},

$$\partial_x^{-1} f(x) = \int_{-\infty}^{x} f(y)\, dy,$$

where $f \in L_x^1(\mathbb{R})$.

The dispersive Hunter-Saxton equation arises as a perturbation of the Hunter-Saxton equation

$$u_t + uu_x = \frac{1}{2}\partial_x^{-1}(u_x^2), \tag{1.3.2}$$

which was derived in [73] as an asymptotic model for the formation of nematic liquid crystals under a director field. It turns out that (1.3.2) is completely integrable [79, 18] with a bi-Hamiltonian structure [118]. For a detailed discussion of known results, we refer the reader to Section 5.

The dispersive Hunter-Saxton equation (1.3.1) was introduced in [80] as a dispersive regularization of (1.3.2). As for complete integrability, that was observed in [52].

In [5] and in this chapter, we initiate the study of the well-posedness for the dispersive Hunter-Saxton equation (1.3.1). Let

$$X^s = L_x^\infty \cap \dot{H}_x^1 \cap \dot{H}_x^{1+s},$$

where $s \in [0, 1]$. We shall also use the notation $X = X^1$. Our first pair of results concerns the well-posedness of (1.3.1) in X. We begin with local well-posedness:

Theorem 1.3.1 (Ai, A.[5])**.** The dispersive Hunter-Saxton equation (1.3.1) is locally well-posed in X. Precisely, for every $R > 0$, there exists $T = T(R) > 0$ such that for every $u_0 \in X$ with $\|u_0\|_X < R$, the Cauchy problem (1.3.1) has a unique solution $u \in C([0, T], X)$. Moreover, the solution map $u_0 \mapsto u$ from X to $C([0, T], X)$ is continuous.

One key difficulty is that the forcing term for both the dispersive and non-dispersive versions of the Hunter-Saxton equation, the nonlinear term $\frac{1}{2}\partial_x^{-1}(u_x^2)$ is unbounded in any L^p space if $p < \infty$, and in particular, in L^2. This means that we are restricted to only pointwise L^∞ control on u,

Moreover, even though there is a KdV-like dispersive term in (1.3.1), we are not able to use dispersive tools, including local smoothing estimates, due to the fact that the potential in front of the nonlinearity on the left-hand side of (1.3.1) lacks spatial decay. In particular, despite the presence of the aforementioned dispersive term, (1.3.1) exhibits quasilinear behavior. In particular, we strongly expect the solutions to exhibit only continuous dependence on initial data.

However, the dispersive term in (1.3.1) will still influence the behavior of the solutions. To be more precise, our second result states that, in contrast to the non-dispersive case (1.3.2), in which blow up can occur, the KdV-like term ensures that the problem is globally well-posed:

Theorem 1.3.2 (Ai,A.[5])**.** The Cauchy problem (1.3.1) is globally well-posed in X. Moreover, for every $t \geq 0$, we have the global in time bounds

$$\|u(t)\|_{L^\infty_x} \lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}),$$
$$\|u(t)\|^2_{\dot{H}^2_x} \lesssim \|u_0\|^2_{\dot{H}^2_x} + \|u_0\|_{X^0} E_1 + t(E_1 + E_1^{1/2})E_1,$$

where $E_1 = \|u_0\|^2_{\dot{H}^1_x}$ is given by the first conserved energy (1.3.3) below.

We also note that the L^∞ estimate is true even for solutions which are only in X^0.

We proved Theorem 1.3.1 by employing a bounded iterative scheme that analyzes the high and low frequency components separately, as well as frequency envelopes, which were introduced by Tao in [135], the latter being used to prove continuous dependence on the initial data. A systematic presentation of the use of frequency envelopes in the study of local well-posedness theory for quasilinear problems can be found in the expository paper [87], which we broadly follow in the proof of Theorem 1.3.1.

Theorem 1.3.2 is a consequence of the existence of the conserved quantities $E_1(t)$ and $E_2(t)$ under the evolution of (1.3.1), which are explicitly defined as

$$\begin{aligned} E_1(t) &= \int_{\mathbb{R}} u_x(t)^2 \, dx, \\ E_2(t) &= \int_{\mathbb{R}} u_{xx}(t)^2 - u(t) u_x(t)^2 \, dx. \end{aligned} \tag{1.3.3}$$

By using the X^1 well-posedness of Theorems 1.3.1 and 1.3.2 we extend the well-posedness results to lower regularity initial data:

Theorem 1.3.3 (Ai,A.[5])**.** For each $s \in (\frac{1}{2}, 1)$, the Cauchy problem (1.3.1) is locally well-posed in X^s.

In our proof, we rely on obtaining estimates for differences of solutions, which in turn enabled us to use Theorem 1.3.1 as leverage in order to construct X^s solutions as limits of sequences of smooth solutions. We obtained the crucial estimates for differences of solutions by analyzing the linearized equation corresponding to (1.3.1):

$$w_t + (uw)_x + w_{xxx} = \partial_x^{-1}(u_x w_x). \tag{1.3.4}$$

Theorem 1.3.4 (Ai,A.[5])**.** For each $s \in (\frac{1}{2}, 1)$, the Cauchy problem (1.3.1) is globally well-posed in X^s. Moreover, for every $t \geq 0$,

$$\begin{aligned} \|u(t)\|_{L^\infty_x} &\lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}) \\ \|u(t)\|^2_{\dot{H}^{1+s}_x} &\lesssim \langle t\rangle^4 E_1^2 \langle \|u_0\|_{X^0} + E_1\rangle^2 + \|u_0\|^2_{\dot{H}^{1+s}_x}, \end{aligned} \tag{1.3.5}$$

where $E_1 = \|u_0\|^2_{\dot{H}^1_x}$.

Our proof of Theorem 1.3.4 relied on using the quadratic normal form transformation associated to (1.3.1) in order to construct modified energy functionals which are norm-equivalent to the $\dot{H}^{1+s}$ energy. For the KdV equation, Christ-Colliander-Tao [30] employed a generalized Miura transform in order to construct $H^{-3/4}$ solutions. In what concerns the Benjamin-Ono equation, Ifrim-Tataru [85] used a partial normal form transform, combined with an exponential renormalization in order to construct solutions in L^2.

Chapter 2

Gallery waves for the vacuum irrotational compressible Euler equations

2.1 Introduction

In this chapter we are concerned with the evolution for the free boundary problem in a compressible gas setting. In one of the most common models, the gas occupies a domain Ω_t with boundary Γ_t at a given time t, and is described by the (Eulerian) variables (ρ, v), where $\rho \geq 0$ is the density, and v is the velocity. The evolution is described by what are known as the compressible Euler equations

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0 \\ \rho(v_t + (v \cdot \nabla)v) + \nabla p = 0 \end{cases} \tag{2.1.1}$$

In the irrotational case, we shall momentarily see that the velocity admits a potential which solves a wave equation with respect to the acoustic metric, and the same will be true for the linearized counterpart of the Euler system. An interesting question here is to which extent Strichartz estimates hold for the linearized problem, as well as for the full nonlinear equations. Our paper is a first step towards understanding the first part of this question in a model case. Specifically, it turns out that the geometry generated by the resulting acoustic metric gives rise to multiply reflecting geodesics along the boundary, which in turn allow us to construct solutions that force derivative losses in the estimates.

We shall assume that the pressure is given by constitutive laws of the form

$$p(\rho) = \rho^{\kappa+1}, \kappa > 0. \tag{2.1.2}$$

This system can be regarded as a coupled one consisting of a wave equation for the variables $(\rho, \nabla \cdot v)$, and a transport equation for the *vorticity* $\omega = \operatorname{curl} v$. Throughout this chapter, we shall assume that our gas is irrotational, so that $\omega = 0$, and $v = \nabla\phi$, where ϕ is a potential. This will allow us to work in a slightly different interpretation, one in which ϕ also solves a wave equation, which will be derived in Section 2.6, along with a linearized counterpart. In both interpretations, a key quantity is the *speed of sound* c_s, which is the propagation speed for the wave components, and is given by

$$c_s^2 = p'(\rho). \tag{2.1.3}$$

In our model, we shall allow the density ρ to vanish, a scenario which corresponds to *vacuum states*, so that the gas will occupy the domain $\Omega_t = \{(t,x)|\rho(t,x) > 0\}$, with a moving boundary Γ_t. In contrast to the fluid case, the density will vanish on the free boundary Γ_t, which means that it can be described as

$$\Gamma_t = \{(t,x)|\rho(t,x) = 0\}.$$

In general, one expects a single stable nontrivial physical regime, known as *physical vacuum*, which corresponds to the situation where $d(x,\Gamma_t) \approx c_s^2(t,x)$. Heuristically, if the sound speed has a faster decay rate, one expects the particles on the boundary to move linearly and independently, which is a regime that can only last for a short time. A slow decay rate for the speed of sound would correspond to an infinite initial acceleration of the boundary, which is another regime that cannot last for a long time. On the other hand, the physical vacuum regime allows the free boundary to move with a bounded velocity and acceleration while interacting with the interior, ensuring that linear waves with speed c_s can reach the free boundary Γ_t in finite time.

Historically, two main approaches have been adopted in fluid dynamics: an Eulerian one, in which the reference frame is fixed, and the particles are moving, and a Lagrangian one, in which the converse is true, in the sense that the particles are stationary, while the frame is moving. Both of these interpretations have been broadly used and studied in the context of the compressible Euler equations in the full space $\mathbb{R}^d$, where the local well-posedness problem is well-studied and understood.

For example, in the full space $\mathbb{R}^d$, the compressible Euler equations have been traditionally regarded as a symmetric hyperbolic system, which means that the results of Hughes-Kato-Marsden from [68] apply here (see also Majda's work in [113]). It follows that the problem is locally well-posed in H^s in the Hadamard sense, where $s > \frac{d}{2} + 1$, with the continuation criterion

$$\int_0^T \|\nabla(\rho, v)(t)\|_{L^\infty}\, dt < \infty$$

In the irrotational case, the general Strichartz estimates of Smith-Tataru from [131] apply directly, which allows one to improve the local well-posedness regularity threshold to $s > \frac{d+1}{2}$, when $d = 3, 4, 5$. In the rotational case, it is not yet known what would be the right condition to impose on the vorticity that would guarantee a similar result; see the results of Disconzi-Luo-Mazzone-Speck [48], Wang [143], Andersson-Zhang [147],[13], and Zhang [146].

On the other hand, until recently, the free boundary problem corresponding to the physical vacuum had not been studied in the Eulerian setting, and all results had been obtained in Lagrangian coordinates, at high regularity, and in indirectly defined spaces.

The first ones to develop a fully Eulerian low regularity approach to the well-posedness of the compressible Euler equations in a physical vacuum were Ifrim and Tataru in [88]. In their paper, they:

- proved the uniqueness of solutions, with minimal assumptions on regularity: $(\rho, v) \in \mathrm{Lip}$. They also showed that the solutions are stable, in the sense that the distance between different solutions can be propagated in time;
- developed an Eulerian Sobolev function space structure tailored to this problem;
- proved sharp, scale invariant energy estimates in the aforementioned spaces, while also providing a minimal continuation criterion for the regularity of solutions ($v \in \mathrm{Lip}$);
- constructed regular solutions in the Eulerian framework in high regularity spaces
- developed a nonlinear Littlewood-Paley decomposition, which allowed them to construct rough solutions as limits of smooth solutions, while also proving continuous dependence on the initial data. Their low regularity threshold morally corresponds to the $\frac{d}{2} + 1$ one obtained by Hugh-Kato-Marsden, being one derivative above scaling.

Together with Marcelo Disconzi, they later proved counterparts of these results for the relativistic Euler equations in a vacuum setting in [47].

In the fluid incompressible case, analogous results were obtained by Ifrim, Pineau, Tataru, and Taylor in [83]. One has to note that this problem concerns the fluid cases, which poses different difficulties. For example, the density does not tend to 0 as we approach the boundary. In particular, one must also assume the Taylor sign condition (this is necessary, as proved by Ebin in [50]), and include the Taylor coefficient in the estimates. We must also mention that the first ones to adopt an Eulerian approach for fluid equations in a free boundary setting were Shatah and Zheng in [124, 125, 126]. However, their main focus is on the free boundary Euler equations with surface tension, and even though they also prove the existence of a solution to the pure gravity problem in the zero surface tension limit, their argument seems to rely on the boundedness of the curvature, which would in turn require a higher degree of regularity than in [83]. It is easier to draw comparisons between the latter and the memoir of Wang, Zhang, Zhao, and Zeng [142]. There, they prove the uniqueness and existence of solutions at a level of regularity which is one derivative above scaling, but their approach is restricted to graph domains with unbounded curvature. By comparison, the results of Ifrim-Pineau-Tataru-Taylor from [83] not only apply to more general domains with potentially more complicated geometries, but they also obtained the first proof of continuity of solutions with respect to the initial data, along with an enhanced uniqueness result, a construction of rough solutions as unique limits of smooth ones, refined low regularity energy estimates with pointwise geometric control parameters that only require very limited regularity, a new proof of the existence of smooth solutions, and an essentially scale invariant continuation criterion, akin to the one obtained by Beale-Kato-Majda for the incompressible Euler equations on the whole space. Therefore, their approach is very different from the one in [142] which in turn relies on the one adopted in the works of Alazard-Burq-Zuily [9, 10]. See also de Poyferré [44].

In this chapter, we shall consider the irrotational compressible Euler equations in a vacuum regime, in the Eulerian setting.

2.1.1 Notations and the conserved energy

The material derivative D_t is defined as the derivative along the particle flow, and is given by the relation

$$D_t = \partial_t + v \cdot \nabla.$$

The equations (2.1.1) can be rewritten as

$$\begin{cases} D_t\rho + \rho\nabla \cdot v = 0 \\ \rho D_t v + \nabla p = 0 \end{cases} \tag{2.1.4}$$

If we differentiate the equation for ρ once again, we obtain a wave equation

$$D_t^2 - \rho\nabla \cdot (\rho^{-1}p'(\rho)\nabla\rho) = \rho[(\nabla \cdot v)^2 - \mathrm{Tr}(\nabla v(\nabla v)^T)]$$

with propagation speed c_s. One can obtain a similar equation for $\nabla \cdot v$.

For the vorticity $\omega = \operatorname{curl} v$, one can obtain the transport equation

$$D_t\omega = -\omega(\nabla v) - (\nabla v)^T\omega.$$

These show that the compressible Euler equations can indeed be interpreted as a coupled system consisting of a pair of variables $(\rho, \nabla \cdot v)$ which solve a wave equation, and a transport equation for the vorticity $\omega = \operatorname{curl} v$. However, we note that the irrotationality condition will allow us to derive a different formulation for our problem, in terms of a stream function. We are going to discuss this in detail in Section 3.

The equations admit a conserved energy, given by

$$E = \int_{\Omega_t} \frac{1}{2}\rho|v|^2 + \rho h(\rho)\, dx,$$

where h is the specific enthalpy, and is defined by

$$h(\rho) = \int_0^\rho \frac{p(\lambda)}{\lambda^2}\, d\lambda.$$

In a suitable setting, this energy can be interpreted as a Hamiltonian, see Chorin-Marsden [29], Ebin-Marsden [51], and Marsden-Ratiu-Weinstein [115].

2.1.2 The good variables

Ifrim and Tataru recast the equation using a pair of variables that turned out to be more convenient to use. We shall briefly recall their justification.

It is not difficult to see that when $\kappa = 1$, the variables (ρ, v) are quite convenient to use. However, we can make a better choice when $\kappa \neq 1$. In order to see this, by using the constitutive law, we get that the sound speed is given by $c_s^2 = (\kappa+1)\rho^\kappa$. We would like this to have linear behavior towards the boundary and exhibit decay, which shows that using $r = r(\rho)$ given by $r' = \rho^{-1}p'(\rho)$ might be a better choice.

Indeed, this is tantamount to setting $r = \dfrac{\kappa+1}{\kappa}\rho^\kappa$, and it turns out that the equations can indeed be simplified, taking the form

$$\begin{cases} D_t r + \kappa r(\nabla \cdot v) = 0 \\ D_t v + \nabla r = 0 \end{cases} \tag{2.1.5}$$

These are the equations that we shall consider throughout this chapter, coupled with the condition $\operatorname{curl} v = 0$, thus taking the form

$$\begin{cases} D_t r + \kappa r(\nabla \cdot v) = 0 \\ D_t v + \nabla r = 0 \\ \operatorname{curl} v = 0. \end{cases} \tag{2.1.6}$$

We shall also need their linearized counterparts

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa s(\nabla \cdot v) + \kappa r(\nabla \cdot w) = 0 \\ D_t w + (w \cdot \nabla)v + \nabla s = 0, \end{cases} \tag{2.1.7}$$

respectively

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa s(\nabla \cdot v) + \kappa r(\nabla \cdot w) = 0 \\ D_t w + (w \cdot \nabla)v + \nabla s = 0 \\ \operatorname{curl} w = 0. \end{cases} \tag{2.1.8}$$

In the next section, we shall see that they admit stream function formulations, in the sense that here exists a velocity potential ϕ such that

$$\begin{aligned} v &= \nabla\phi \\ D_t\phi &= \frac{|v|^2}{2} - r, \end{aligned} \tag{2.1.9}$$

with ϕ also satisfying

$$D_t^2\phi + \nabla r \cdot \nabla\phi - \kappa r \Delta\phi = 0. \tag{2.1.10}$$

The linearized velocity also admits a potential ψ, which turns out to satisfy the following equations:

$$\begin{aligned} w &= \nabla\psi \\ D_t\psi &= -s. \end{aligned} \tag{2.1.11}$$

Moreover, the linearized velocity potential ψ also satisfies the following acoustic wave equation:

$$D_t^2\psi - \nabla r\cdot\nabla\psi - \kappa r\Delta\psi = \kappa s(\nabla\cdot v). \tag{2.1.12}$$

We note that we do not impose boundary conditions for the linearized equations. One of the reasons stems from the manner in which the linearized variables (s, w) naturally arise. They are usually defined by considering a one-parameter family of solutions to the system (1.1.5), and then by taking

$$(s, w) = \left(\frac{d}{dh}_{|h=h_0} r^h, \frac{d}{dh}_{|h=h_0} v^h\right),$$

for a fixed value h_0. Now, when h varies, the domain of the solution (r^h, v^h), which is defined as $\{r^h > 0\}$, will also vary, and this precludes us from being able to impose boundary conditions for the linearized system in a meaningful way.

Another reason is that for $\kappa < 1$, when we require the solutions to the previous acoustic wave equation for the linearized velocity potential ψ to belong to the energy space $\mathscr{H}$, which will be defined in the next subsection, this condition will already encapsulate significant information, and we would otherwise risk ending up with only trivial solutions. For a more detailed discussion of this aspect, see Subsection 2.5.1.

The previous acoustic wave equation also appears in helioseismology, describes solar oscillations, and is a simplified scalar equation introduced by Gizon-Barucq-Duruflé-Hanson-Leguèbe-Birch-Chabassier-Fournier-Hohage-Papini in [62]; see also [2, 1, 117], as well as Muller's PhD thesis for results on the uniqueness of the associated inverse problem under various hypotheses. Even though this is a simplified version of the equation of stellar oscillations (see [112] and [99]) and Galbrun's equation, the latter being introduced in [53], it still captures a large proportion of the solar dynamics; see also [45, 81, 19], as well as [65, 66, 64]. Historically, there have been two approaches in the field of seismology: a global one, in which one solves inverse spectral problems, in order to be able "hear" the shape of the Sun (one of the results of this approach is the radially symmetric model, see [32, 31]), and a local one, which relies on data such as travel times and correlations of oscillations, as well as on solving inverse boundary problems in order to construct a 2D or 3D image of the solar interior (see [49, 61]). Applications of the field of helioseismology include space weather prediction, and understanding the solar cycle.

2.1.3 Function spaces and energies

The conserved energy takes the form

$$E = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} \left(r^2 + \frac{\kappa+1}{2} r v^2 \right) dx$$

This motivated Ifrim and Tataru to introduce the base energy space $\mathscr{H}$ with norm

$$\|(s,w)\|_{\mathscr{H}}^2 = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} \left(s^2 + \kappa r w^2 \right) dx,$$

which is associated to the linearized equations (2.1.7) for functions (s,w) defined almost everywhere in the gas domain Ω_t.

For higher regularity spaces, they were motivated by the second order wave equation, whose leading order operator $D_t^2 - \kappa r \Delta$ is naturally associated to the acoustic metric

$$g = r^{-1} dx^2$$

This allowed them to define the higher order Sobolev spaces $\mathscr{H}^{2k}$ for elements $(s,w) \in \mathscr{D}'(\Omega_t) \times \mathscr{D}'(\Omega_t)$, with norm

$$\|(s,w)\|_{\mathscr{H}^{2k}}^2 = \sum_{|\beta| \le 2k}^{|\beta|-\alpha \le k} \|r^\alpha \partial^\beta (s,w)\|_{\mathscr{H}}^2,$$

where $0 \le \alpha \le k$. For fractional k, the spaces $\mathscr{H}^{2k}$ can be defined by interpolation. See [88] for the precise definition and a detailed discussion.

Our problem admits the following scaling law:

$$\begin{aligned} r_\lambda(t,x) &= \lambda^{-2} r(\lambda t, \lambda^2 x) \\ v_\lambda(t,x) &= \lambda^{-1} v(\lambda t, \lambda^2 x), \end{aligned}$$

in the sense that if (r,v) is a solution, then so is (r_λ, v_λ).

We note that the critical exponent k for which the homogeneous counterpart of $\mathscr{H}^{2k}$ is invariant to scaling is

$$2k_0 = d + 1 + \frac{1}{\kappa}$$

In [88], Ifrim and Tataru defined the control parameters

$$A = \|\nabla r - N\|_{L^\infty} + \|v\|_{\dot{C}^{\frac{1}{2}}},$$

which is associated with the critical Sobolev exponent $2k_0$, and

$$B = \|\nabla r\|_{\tilde{C}^{0,\frac{1}{2}}} + \|\nabla v\|_{L^\infty},$$

where

$$\|f\|_{\tilde{C}^{0,\frac{1}{2}}} = \sup_{x,y\in\Omega_t} \frac{|f(x)-f(y)|}{r(x)^{\frac{1}{2}}+r(y)^{\frac{1}{2}}+|x-y|^{\frac{1}{2}}}$$

They also introduced the phase space

$$\mathbf{H}^{2k} = \{(r,v)|(r,v)\in \mathscr{H}^{2k}\}.$$

Due to the fact that the $\mathscr{H}^{2k}$ norm depends directly on Ω_t (hence on r), this should be regarded as an infinite dimensional manifold. Their main local well-posedness result is as follows:

Theorem 2.2. [Ifrim, Tataru,[88]] The system (2.1.5) is locally well-posed in the space $\mathscr{H}^{2k}$ for $k\in\mathbb{R}$ satisfying

$$2k > 2k_0 + 1.$$

One of the key ingredients in their proof consisted of deriving energy estimates of the form

$$\|(r,v)(t)\|_{\mathscr{H}^{2s}} \lesssim e^{\int_0^t C(A)B(u)\,du}\|(r,v)(0)\|_{\mathscr{H}^{2s}}.$$

For the previously defined control parameters, we have the following Sobolev estimates

$$\begin{aligned} A &\lesssim \|(r,v)\|_{\mathbf{H}^{2k}}, k>k_0\\ B &\lesssim \|(r,v)\|_{\mathbf{H}^{2k}}, k>k_0+\frac{1}{2}. \end{aligned}$$

The previous threshold in Theorem 2.2 is required when one tries to control $B(t)$ pointwise in time. However, this quantity appears in the energy estimates in a time-averaged manner, so a natural question is whether one can take advantage of this in order to lower the regularity threshold. This in turn motivates us to investigate Strichartz estimates for our problem.

2.2.1 Main results

For the rest of this chapter, we shall consider the simplified model

$$\Omega = \{x\in\mathbb{R}^d | x_d > 0\},$$

with $r = x_d$, and $v = 0$. The corresponding wave equation satisfied by the linearized potential ψ is

$$\partial_t^2\psi - \kappa x_d\Delta\psi - \partial_d\psi = 0. \tag{2.2.1}$$

In what follows, for a given vector $a\in\mathbb{R}^d$, we shall denote by a' its tangential coordinates, corresponding to the first $d-1$ coordinates, and we also consider $g = \kappa^{-1}x_d^{-1}dx^2$, which is the acoustic metric.

Definition 2.3. Let $d, q, r \geq 2$, $\gamma \in \mathbb{R}$.

1. We say that (q, r) is *wave-admissible* if
$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}.$$
If we have equality in the previous inequality, we say that (q, r) is *sharp wave-admissible.*

2. We say that (q, r, γ) is a wave Strichartz triple if (q, r) is wave-admissible and
$$\frac{1}{q} + \frac{d}{2r} = \frac{d}{2} - \gamma.$$

For reference, recall the Strichartz estimates for the wave equations in $\mathbb{R}^d$:

Theorem 2.3.1. Let $d \geq 2$, (q, r, γ) be wave-admissible, and let u be a solution to the initial value problem
$$\begin{aligned}(\partial_t^2 - \Delta)u &= 0\\ (u, u_t)(0) &= (u_0, u_1).\end{aligned}$$
Then,
$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{\dot{H}_x^\gamma(\mathbb{R}^d)} + \|u_1\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^d)}.$$

We also define the Littlewood-Paley tangential projectors:

Definition 2.4. $P_{x',\lambda}$ is the Fourier multiplier defined by the relation
$$\widehat{P_{x',\lambda}} f(\xi') = \left(\psi\left(\frac{\xi'}{\lambda}\right) - \psi\left(\frac{2\xi'}{\lambda}\right)\right)\hat{f}(\xi'),$$
where ψ is a radial smooth function supported in the region $\{\xi' \in \mathbb{R}^{d-1} | \|\xi'\| \leq 2\}$, with $\psi = 1$ when $\{\xi' \in \mathbb{R}^{d-1} | \|\xi'\| \leq 1\}$.

Our first main result shows that these solutions do indeed provide counterexamples to Strichartz estimates in the context of the wave equation. More precisely, we have the following

Theorem 2.4.1. Let (q, r, γ) be wave-admissible in dimension $d \geq 2$. Then, for every $s \in \mathbb{R}$ with $s < \dfrac{1}{q} + \gamma + \dfrac{1}{2\kappa} + 1$, there exists a sequence of solutions $(\psi^j)_{j\geq 0}$ to the initial value problem
$$\begin{aligned}(\partial_t^2 - \kappa x_d \Delta - \partial_d)\psi^j &= 0\\ (\psi^j, \partial_t \psi^j)(0) &= (\psi_0^j, \psi_1^j),\end{aligned}$$

with $(P_{x',2^{2j}}\psi_0^j, P_{x',2^{2j}}\psi_1^j) = (\psi_0^j, \psi_1^j)$, such that

$$\sup_{j\geq 0}(\|(\psi_1^j, \nabla_x \psi_0^j)\|_{\mathscr{H}^{2s}}) \leq 1,$$

and whenever $\alpha \in \left(0, \frac{1}{q} + \gamma + \frac{1}{2\kappa} + 1 - s\right)$, we have

$$\lim_{j\to\infty} 2^{-2j\alpha}\|\nabla^2\psi^j(t,\cdot)\|_{L_t^q L_x^r([0,1]\times\Omega)} = \infty.$$

We recall that our problem admits a scaling symmetry, which for the potential ψ takes the form

$$\psi_\lambda(t,x) = \lambda^{-3}\psi(\lambda t, \lambda^2 x)$$

In what follows, we show that our problem doesn't even admit Strichartz estimates corresponding to its own intrinsic scaling. In order to see this, we note that a Strichartz estimate for our problem would need to have the form

$$\||\nabla_x|^\gamma \psi\|_{L_t^q L_x^r([0,1]\times\Omega)} \lesssim \|(\partial_t\psi(0), \nabla_x\psi(0))\|_{\mathscr{H}},$$

which, upon imposing the scaling invariance, motivates the following

Definition 2.5. Let $d, q, r \geq 2$, $\kappa > 0$, and $\gamma \in \mathbb{R}$. We say that (q, r, γ) is an *Euler Strcihartz* triple if (q, r) is wave-admissible and

$$\frac{1}{2q} + \frac{d}{r} = \frac{d}{2} + \frac{1}{2\kappa} + \gamma - 1.$$

Our second main result is the following

Theorem 2.5.1. Let (q, r, γ) be an Euler Strichartz triple in dimension $d \geq 2$. Then, for every $s \in \mathbb{R}$ with $2s < \frac{1}{q} - 2\gamma$, there exists a sequence of solutions $(\psi^j)_{j\geq 0}$ to the initial value problem

$$(\partial_t^2 - \kappa x_d\Delta - \partial_d)\psi^j = 0$$
$$(\psi^j, \partial_t\psi^j)(0) = (\psi_0^j, \psi_1^j),$$

with $(P_{x',2^{2j}}\psi_0^j, P_{x',2^{2j}}\psi_1^j) = (\psi_0^j, \psi_1^j)$, such that

$$\sup_{j\geq 0}(\|(\psi_1^j, \nabla_x \psi_0^j)\|_{\mathscr{H}^{2s}}) \leq 1,$$

and whenever $\alpha \in \left(0, \frac{1}{2q} - \gamma - s\right)$, we have

$$\lim_{j\to\infty} 2^{-2j\alpha}\|\nabla^2\psi^j(t,\cdot)\|_{L_t^q L_x^r([0,1]\times\Omega)} = \infty.$$

Remark 2.5.1. In particular, Theorems 2.4.1 and 2.5.1 imply that when $(q, r) = (2, \infty)$, we deduce that for every $s \in \mathbb{R}$ with $2s < 2k_0 + 1$, there exists a sequence of solutions $(\psi^j)_{j \geq 0}$ to the initial value problem

$$(\partial_t^2 - \kappa x_d \Delta - \partial_d)\psi^j = 0$$
$$(\psi^j, \partial_t \psi^j)(0) = (\psi_0^j, \psi_1^j),$$

with $(P_{x',2^{2j}}\psi_0^j, P_{x',2^{2j}}\psi_1^j) = (\psi_0^j, \psi_1^j)$, such that

$$\sup_{j \geq 0}(\|(\psi_1^j, \nabla_x \psi_0^j)\|_{\mathscr{H}^{2s}}) \leq 1,$$

and whenever $\alpha \in \left(0, k_0 + \frac{1}{2} - s\right)$, we have

$$\lim_{j \to \infty} 2^{-2j\alpha} \|\nabla^2 \psi^j(t, \cdot)\|_{L_t^2 L_x^\infty([0,1]\times\Omega)} = \infty.$$

The key idea behind the proofs of Theorems 2.4.1 and 2.5.1 is that for every given frequency, there exist geodesics with multiple periodic reflections next to the boundary, and exact periodic solutions to the acoustic wave equations that are highly localized in tangential frequency and propagate along them. It also turns out that there exist similar solutions to the latter, which are still highly localized in tangential frequency and propagate along the aforementioned geodesics, at least up until a certain time of coherence. In analogy with Ivanovici [90], we shall call them *gallery waves* or *gallery modes*. Of course, it is natural to also investigate the behavior of gallery waves for individual frequencies and modes. Our result is as follows:

Theorem 2.5.2. Let $d \geq 2$, and u_0 a gallery wave corresponding to the mode μ, where a precise definition is going to given in Section 2.9. Then, the solution u to the initial value problem

$$(\partial_t^2 - \kappa x_d \Delta - \partial_d)u = 0$$
$$(u, u_t)(0) = (u_0, 0),$$

where $u_0 = P_{x',2^{2j}}u_0$, we have

$$\|u\|_{L_t^q L_x^r([0,T_0]\times\Omega)} \lesssim \left(2^{2j}\right)^{\left(\frac{3d+1}{4}\right)\left(\frac{1}{2}-\frac{1}{r}\right)+\frac{1}{2\kappa}-1} \|(0, \nabla_x u_0)\|_{\mathscr{H}(\Omega)},$$

whenever

$$\frac{1}{q} = \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right),$$

with $q, r \geq 2$.

One might expect to have this type of gallery waves solutions for the fully nonlinear equations as well. Another interesting question is to study gallery modes for the fully nonlinear equation, and to also analyze their nonlinear self-interactions. Further questions to be considered are whether stationary solutions are stable upon perturbations with gallery modes, how two gallery modes could interact, and up to which time they are coherent.

2.5.1 A heuristic description of gallery waves

The behavior that we expect our gallery waves to have is consistent with the behavior of multiply reflecting geodesics on the boundary. It turns out that this phenomenon also characterizes the ray acoustics in the interior of the Sun, as noted in [60].

By applying the time Fourier transform to equation (2.2.1), we get

$$(-\kappa x_d \Delta'_x - \kappa x_d \partial_d^2 - \partial_d - \tau^2)\tilde{\psi} = 0$$

Let ξ' be the tangential frequency, and ξ_d the normal frequency. Upon taking the Fourier transform in x', we get

$$(-\kappa x_d \partial_d^2 - \partial_d + \kappa x_d |\xi'|^2 - \tau^2)\tilde{\psi} = 0.$$

Heuristically, when $x_d \ll \tau^{-2}$, the uncertainty principle suggests that $\delta\xi_d \gg \tau^2$. Therefore, once we also restrict to $|\xi_d| \gtrsim |\xi'|$, we would have

$$(-\kappa x_d \partial_d^2 - \partial_d)\tilde{\psi} \approx 0$$

This equation has two homogeneous solutions, $\tilde{\psi} = x_d^{\frac{1}{\kappa}-1}$ and $\tilde{\psi} = 1$; assuming that $\kappa < 1$, the finiteness of the energy

$$\int x_d^{\frac{1}{\kappa}-1}(s^2 + \kappa x_d |w|^2)\, dx$$

implies that only $\tilde{\psi} = 1$ is viable. In particular, this would correspond to a solution which is roughly constant in the normal variable in the region $x_d \lesssim \tau^{-2}$. This resulting property of the solution is another reason why we do not impose boundary conditions for the linearized acoustic equation.

Another interesting regime occurs when $x_d|\xi'|^2 \gg \tau^2$. In this case, the elliptic effect will dominate in our equation for $\tilde{\psi}$, leading in particular to a very quick decay as x_d grows.

We expect these two regimes to overlap when $\tau^2 = x_d|\xi_d|^2$ and $\tau^2 = x_d|\xi'|^2$. Fixing $x_d \approx 2^{-2j}$, the uncertainty principle implies $|\tau| \approx 2^j$, $|\xi'| \approx 2^{2j}$, and $|\xi_d| \approx 2^{2j}$. These correspond to a solution akin to a gallery mode, behaving like a bump function in both the normal and the tangential directions, and roughly periodic in time, with small variations and an oscillation period roughly equal to 2^{-j}. This suggests that Ifrim and Tataru's result, Theorem 2.2 from [88], might be optimal at least in the frequency range where $\tau^2 \lesssim |\xi'|$. However, this leaves open the question of improving this result away from this range.

Classically, gallery modes arise as highly localized waves traveling along the boundary in convex domains in the study of the wave and Schrödinger equations. They were studied by Ivanovici in [90] in the context of finding counterexamples for Strichartz estimates in strictly convex domains with boundary; see also [91, 93, 92, 94, 95, 23].

2.5.2 Further historical comments

The compressible Euler equations have been studied for a long period of time, and have also received a great deal of interest from the physical side. Allowing the density to vanish in some regions, a regime which corresponds to vacuum states, adds significantly many layers of difficulty to the problem, due to the fact that the behavior of the gas is heavily influenced by the speed of sound at the boundary. Moreover, physical vacuum is also the natural boundary condition for compressible gases. Here, one should regard the space as being divided into a particle region Ω_t, and a vacuum region, which are separated by a free boundary $\Gamma_t = \partial\Omega_t$ that evolves in time. One can identify two main scenarios here, depending on the behavior of the density, or equivalently, of the sound speed c_s at the free boundary:

- A fluid case, in which the density and the sound speed are assumed to have a nonzero positive limit at the free boundary.
- A gas scenario, in which the density tends to zero near the free boundary, which is going to be the case that we shall focus on in this chapter.

Fluid flows were investigated in Christodoulou-Miao [33] and Lindblad [107], while the incompressible case was also studied by Lindblad-Luo [108]. In the incompressible case, we once again note the more recent results of Ifrim-Pineau-Tataru-Taylor [83] that we have already discussed in this introduction.

Based on the relation between the speed of sound and the distance to the boundary of a given particle, one can distinguish three cases:

a) A fast decay scenario, in which $c_s \lesssim d_{\Gamma_t}$. Here, the vacuum boundary is going to be expected to evolve linearly, preventing internal waves from reaching the boundary arbitrarily fast. This means that this scenario will survive for at least a short amount of time, and that one can also apply the known results on symmetric hyperbolic systems in order to analyze the local well-posedness of the problem, see for instance DiPerna [46], Chen [28], and Lions [110], as well as Kawashima-Makino-Ukai [114], Liu-Yang [111], and Chemin [27]. This means that this scenario does not yield a genuine free boundary problem, and as Chemin [27] shows in the one dimensional case, the geometry breaks down in finite time.

b) A slow decay scenario, in which $c_s \gg d_{\Gamma_t}$. In this case, the internal waves can reach the boundary arbitrarily fast, causing the internal flow to be strongly coupled with the one of the free boundary, thus rendering this case a genuine free boundary problem. A natural choice for the decay rates is represented by the family $c_s \approx d_{\Gamma_t}^{\beta}$, where $\beta \in (0, 1)$. Among all of these, physical and mathematical considerations suggest that there exists a single stable decay rate, corresponding to $\beta = \frac{1}{2}$. The other cases are conjectured to be unstable and to instantly transition into the stable regime. However, no rigorous results have been proved in this sense, and it is also highly likely that the scenarios $\beta < \frac{1}{2}$ and $\beta > \frac{1}{2}$ differ significantly.

In the physical vacuum case, the first setting that was historically considered was the one dimensional one in Coutand-Shkoller [42] and Jang-Masmoudi [96]. While Coutand and Shkoller prove some energy estimates and provide a procedure to construct solutions, the function spaces are not completely defined, and the initial data is not directly described. This matter is addressed by Jang and Masmoudi, who introduce the Lagrangian counterparts of Ifrim and Tataru's weighted Sobolev spaces, and prove the existence and uniqueness of solutions in sufficiently regular spaces.

The three dimensional case has recently received significant attention. In particular, in Coutand-Lindblad-Shkoller [40], energy estimates are formally derived in the case $\kappa = 1$. In Coutand-Shkoller [41], existence is also proved via a parabolic regularization procedure, but once again, in a functional setting that is not complete; moreover, their difference bound requires more regularity on the solutions than the existence result. Independently, Jang and Masmoudi [97] also use a parabolic regularization to prove the existence and uniqueness for solutions for arbitrary $\kappa > 0$, but with a different proof of the energy estimates. However, they carry out their proofs only on the torus in the Lagrangian setting, only briefly outlining the general case.

All of these results in the vacuum case were proved in the Lagrangian setting, and as we have already mentioned in this introduction, the first fully Eulerian results were proved by Ifrim and Tataru [88], who, together with Disconzi, also proved the counterparts of these results for the relativistic equations [47]. Later on, together with Pineau and Taylor they also obtained counterparts for these results in the incompressible fluid case [83].

Strichartz estimates for the wave and Schrödinger equation have a long history, and they were first derived in the case $q = r$ by Strichartz in [132] for the wave and classical Schrödinger equations. These results were later extended to mixed $L_t^q L_x^r$ by Ginibre and Velo [58] for the Schrödinger equation, and then independently by Ginibre-Velo [59] and Lindblad-Sogge [109], following earlier work by Kapitanskii [101]. The remaining endpoints were settled by Keel-Tao [103].

In the context of wave equations with smooth variable coefficients, the first results were obtained independently by Mockenhaupt-Seeger-Sogge [116] and Kapitanskii [100].For rough coefficients, the first results were obtained in dimensions $d = 2$ and $d = 3$ by Smith [127] for metrics of class C^2 by using wave packet techniques. Smith and Sogge also showed that when the metric has Hölder regularity C^α, with $\alpha < 2$, the result might fail.

The first improvements were obtained independently by Bahouri-Chemin [16] and [138], in which they obtained Strichartz estimates with a $\frac{1}{4}$-derivative loss. These results were improved by Tataru in all dimensions in [139] and [140], where he obtained Strichartz estimates with a $\frac{1}{6}$-derivative loss. In the meantime, Bahouri and Chemin also improved their own results in [15] to a loss of derivatives slightly better than $\frac{1}{5}$. However, Smith and Tataru subsequently proved in [130] that for general metrics of class C^1, a loss of $\frac{1}{6}$ is optimal. In the case of quasilinear wave equations, lossless Strichartz were obtained by by Smith and Tataru [131] in dimensions $d = 2$ and $d = 3$.

For manifolds with smooth and strictly geodesically concave boundary, Smith and Sogge [128] used the Melrose and Taylor parametrix to prove the Strichartz estimates in the non-endpoint cases for the corresponding wave equation. Here, the concavity condition is essential, as in its absence, there could exist multiply reflecting geodesics, as well as their limits consisting of gliding rays, which would prevent the existence of the aforementioned parametrix.

Koch, Smith, and Tataru [106] proved "log-loss" estimates for the spectral clusters on compact manifolds without boundary. Burq, Lebeau, and Planchon [26] obtained Strichartz type inequalities on manifolds with boundaries by using the $L^r(\Omega)$ estimates that had been proved by Smith and Sogge [129] for a class of spectral operators. However, the range of indices (q, r) is restricted by the admissible range for r in the squarefunction estimate fort he wave equation, which controls u in the space $L^r(\Omega, L^2((-T, T))$. For example, when $d = 3$, this restricts (q, r) to $q, r \geq 5$. Blair, Smith and Sogge [23] later extended the result from [26], proving that if Ω is a compact manifold with boundary, and (q, r, β) a triple with

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta,$$

subject to

$$\frac{3}{r} + \frac{d-1}{r} \leq \frac{d-1}{2}, \text{ when } d \leq 4, \text{ and } \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, \text{ when } d \leq 4,$$

then Strichartz estimates hold for solutions to the wave equations with homogeneous Dirichlet or Neumann boundary conditions, with the implicit constant depending solely on Ω and T.

In the context of strictly convex manifolds with boundary, counterexamples to Strichartz estimates and the extent to which they still hold were studied by Ivanovici in [90] in the case of the upper half-space. In the Friedlander model case, dispersive estimates for the general wave equations were investigated by Ivanovici, Lebeau, and Planchon in [93]; for large time dispersive estimates for the Klein-Gordon and wave equations, see [91]. For the general case, see the works of Ivanovici, Lebeau, and Planchon, or with Lascar [92, 94]. For counterexamples to Strichartz estimates in two dimensional convex domains, see another work of the same authors [95].

2.5.3 An outline of the chapter

Here, we briefly describe the main structure of this chapter, along with some of the main ideas in each section.

Velocity potential formulation for the equation

In this short section we take advantage of the irrotationality of v and of its linearized counterpart w in order to derive simpler, stream function formulations for both the fully nonlinear equations and the linearized one. In particular, it turns out that both stream functions solve wave equations.

The Hamiltonian of the problem and its bicharacteristics

The goal of this section is to determine the bicharacteristics of the Hamiltonian flow associated to our wave operator $\partial_t^2 - \kappa x_d \Delta - \partial_d$, which will also provide us with the explicit form of the geodesics associated to the acoustic metric. This will allow us to see that there exist multiply reflecting geodesics next to the boundary, which shall confirm our heuristics. The explicit form of the geodesics will also enable us to motivate the choice of scales in the construction of our gallery, which are the objects of interest in this chapter.

Whispering gallery modes

In this section we shall study the eigenvalue problem associated to our operator at a fixed tangential frequency ξ'. We first study the corresponding eigenfunctions heuristically by performing a WKB analysis. It turns out that the resulting ordinary differential equation has a turning point, where the behavior of the resulting solutions changes drastically. The resulting asymptotics also allow us to determine some relevant limit conditions at the boundary that will allow us to work in our energy space $\mathscr{H}$. We also describe them explicitly in terms of Laguerre polynomials and of hypergeometric functions. Our analysis here will allow us to define our gallery waves.

Proof of Theorem 2.5.2

In this section, we reduce the proof of Theorem 2.5.2 to proving Strichartz estimates for an initial value problem in $d-1$ dimensions. We prove the aforementioned estimates by applying a stationary phase argument in order to derive dispersive estimates, from which the Strichartz estimates immediately follow using the TT^* lemma.

Construction of our counterexamples to the Strichartz estimates

In this section, we construct the functions that are going to serve as our counterexamples to the classical Strichartz estimates as superpositions of wave packets, which are in turn "averages" of gallery modes. Their behavior is going to roughly mimic the behavior of multiply reflecting geodesics.

Proof of Theorems 2.4.1 and 2.5.1

In this section, we use the solutions constructed in Section in order to prove Theorems 2.4.1 and 2.5.1. We deduce that we have a loss in the Strichartz estimates for our equation.

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2.6 Velocity potential formulation for the equations

In this section, we shall take advantage of the irrotationality conditions in equations (2.1.6) and (2.1.8), and derive their stream function formulations.

We recall our equations:

$$\begin{cases} D_t r + \kappa r(\nabla \cdot v) = 0 \\ D_t v + \nabla r = 0 \\ \operatorname{curl} v = 0, \end{cases}$$

as well as the linearized counterparts:

$$\begin{cases} D_t s + w \cdot \nabla r + \kappa s(\nabla \cdot v) + \kappa r(\nabla \cdot w) = 0 \\ D_t w + (w \cdot \nabla)v + \nabla s = 0 \\ \operatorname{curl} w = 0. \end{cases}$$

We claim that they admit the following reformulation:

Proposition 2.7. There exists a potential function ϕ such that

$$\begin{aligned} &D_t \phi = \frac{|v|^2}{2} - r \\ &D_t^2 \phi + \nabla r \cdot \nabla \phi - \kappa r \Delta \phi = 0. \end{aligned}$$

Similarly, for the linearized equation there exists a velocity potential ψ such that

$$\begin{aligned} &D_t \psi = -s \\ &D_t^2 \psi - \nabla r \cdot \nabla \psi - \kappa r \Delta \psi = \kappa s(\nabla \cdot v), \end{aligned}$$

Proof. As v is irrotational, there exists a potential ϕ such that $v = \nabla \phi$. In this case, the equation for v successively becomes:

$$D_t v = -\nabla r \iff D_t(\nabla \phi) = -\nabla r \iff \nabla(D_t \phi) - \frac{1}{2}\nabla|\nabla \phi|^2 = -\nabla r$$
$$\nabla(D_t \phi) = \nabla\left(\frac{1}{2}|\nabla \phi|^2 - r\right) \iff D_t \phi = \frac{1}{2}|\nabla \phi|^2 - r + k(t)$$

By making the substitution

$$\phi \to \phi - \int_0^t k(\tau)\, d\tau,$$

we may assume $k(t) = 0$ for all t. This gives

$$D_t\phi = \frac{|v|^2}{2} - r$$

Upon applying D_t to both sides and using the equations for v and r, it follows that

$$D_t^2\phi = -\nabla\phi \cdot \nabla r + \kappa r \Delta\phi$$

We linearize the potential ϕ in the relation $v = \nabla\phi$, which implies that $w = \nabla\psi$. By linearizing the equality $D_t\phi = \dfrac{|v|^2}{2} - r$, we successively get that

$$\begin{aligned} &D_t\psi + w \cdot v = v \cdot \nabla\psi - s \iff \\ &D_t\psi = -s \end{aligned}$$

By applying D_t to both sides and using the equation for s, we infer that

$$D_t^2\psi - \nabla r \cdot \nabla\psi - \kappa r \Delta\psi = \kappa s(\nabla \cdot v)$$

This finishes the proof. □

As we have already briefly discussed in the introduction and we shall see in more detail in Section 2.9, if we fix a time frequency, in the region $x_d \lesssim \tau^{-2}$, where the uncertainty principle does not yet have a dominating effect, the solution to the equation will asymptotically be equal to a linear combination of two fundamental solutions: $\tilde{\psi} = x_d^{\frac{1}{\kappa}-1}$ and $\tilde{\psi} = 1$. When assuming $\kappa < 1$, the finiteness of the energy

$$\int x_d^{\frac{1}{\kappa}-1}((\partial_t\psi)^2 + \kappa x_d|\nabla\psi|^2)\, dx$$

implies that only $\tilde{\psi} = 1$ is viable. In particular, this would correspond to a solution which is roughly constant in the normal variable in the region $x_d \lesssim \tau^{-2}$. This also shows that we do not impose boundary conditions for the linearized acoustic equation.

2.8 The Hamiltonian of the problem and its bicharacteristics

In this section we shall determine the bicharacteristics of the Hamiltonian flow associated to our wave operator $\partial_t^2 - \kappa x_d \Delta - \partial_d$, which will also provide us with the explicit form of the geodesics associated to the acoustic metric. This will allow us to see that there exist geodesics with multiple periodic reflections next to the boundary, which shall confirm our heuristics. This will also motivate the choice of the scales in our construction of gallery waves, which are going to be our objects of interest.

The Hamiltonian associated to the homogeneous term of second degree of our wave operator has the form

$$H = \kappa x_d \xi_d^2 + \kappa x_d |\xi'|^2 - \tau^2.$$

In what follows, we shall determine its bicharacteristics, which will also give us the geodesics associated to the corresponding acoustic metric.

We consider a bicharacteristic of the form $(t, x_d, x', \tau, \xi_d, \xi')(s)$, where s is the parameter, with initial data $(t_0, x_{d0}, x_0', \tau_0, \xi_{d0}, \xi_0')$. The equations read:

$$\frac{d}{ds}(t, x_d, x', \tau, \xi_d, \xi') = (-2\tau, 2\kappa x_d \xi_d, 2\kappa x_d \xi', 0, -\kappa(\xi_d^2 + |\xi'|^2), 0).$$

It immediately follows that

$$\begin{aligned} \tau(s) &= \tau_0 \\ t(s) &= -2s\tau_0 + t_0 \\ \xi'(s) &= \xi_0'. \end{aligned}$$

When $\xi_0' \neq 0$, we also obtain

$$\begin{aligned} \frac{1}{|\xi'|} \arctan\left(\frac{\xi_d(s)}{|\xi'|}\right) - \frac{1}{|\xi'|} \arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) &= -\kappa s \\ \xi_d(s) &= |\xi'| \tan\left(\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) - \kappa|\xi'|s\right) \end{aligned}$$

This immediately implies that

$$\begin{aligned} x_d(s) &= x_{d0} e^{2\kappa \int_0^s \xi_d(\sigma)\, d\sigma} \\ x_d(s) &= x_{d0} e^{\int_0^s 2\kappa|\xi'| \tan\left(\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) - \kappa|\xi'|\sigma\right) d\sigma} \\ &= x_{d0} e^{2\ln\left|\cos\left(\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) - \kappa|\xi'|s\right)\right| - 2\ln\left|\cos\left(\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right)\right)\right|} \\ &= x_{d0} \frac{\cos^2\left(\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) - \kappa|\xi'|s\right)}{\cos^2\left(\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right)\right)} \\ &= x_{d0} \left(\frac{|\xi|^2}{2|\xi'|^2} + \cos(2\kappa|\xi'|s) \frac{|\xi'|^2 - \xi_{d0}^2}{2|\xi'|^2} + \sin(2\kappa|\xi'|s) \frac{\xi_{d0}}{|\xi'|}\right). \end{aligned}$$

We also have

$$\begin{aligned}\dot{x'} &= 2\kappa\xi_0' x_d \\ &= 2\kappa\xi_0' x_{d0}\left(\frac{|\xi|^2}{2|\xi'|^2} + \cos\left(2\kappa|\xi'|s\right)\frac{|\xi'|^2-\xi_{d0}^2}{2|\xi'|^2} + \sin\left(2\kappa|\xi'|s\right)\frac{\xi_d}{|\xi'|}\right),\end{aligned}$$

hence

$$x'(s) = x_0' + 2\kappa\xi_0' x_{d0}\left(\frac{s|\xi|^2}{2|\xi'|^2} + \frac{\sin\left(2\kappa|\xi'|s\right)}{2\kappa|\xi'|}\frac{|\xi'|^2-\xi_{d0}^2}{2|\xi'|^2} + \frac{1-\cos\left(2\kappa|\xi'|s\right)}{2\kappa|\xi'|}\frac{\xi_{d0}}{|\xi'|}\right)$$

On the other hand, when $\xi_0' = 0$, we have

$$\begin{aligned}\frac{1}{\xi_{d0}} - \frac{1}{\xi_d(s)} &= -\kappa s \\ \xi_d(s) &= \frac{\xi_{d0}}{\kappa s\xi_{d0}+1}.\end{aligned}$$

We immediately obtain

$$\begin{aligned}x_d(s) &= x_{d0}e^{2\kappa\int_0^s \xi_d(\sigma)\,d\sigma} \\ x_d(s) &= x_{d0}e^{\int_0^s \frac{2\kappa\xi_{d0}}{\kappa\sigma\xi_{d0}+1}\,d\sigma} \\ &= x_{d0}e^{2\ln|\kappa s\xi_{d0}+1|} \\ &= x_{d0}\left(\kappa s\xi_{d0}+1\right)^2.\end{aligned}$$

The equation

$$\dot{x'} = 0$$

also implies that

$$x'(s) = x_0'.$$

This fully describes the bicharacteristics of the Hamiltonian flow and the geodesics associated to our acoustic metric as well. Moreover, when $\xi' \neq 0$, we can see that even though our geodesics/bicharacteristics can't be smoothly extended to the boundary, where they develop a cusp singularity, there is one meaningful way in which they can be continued (both forward and backward).

We analyze the case $\xi' \neq 0$, for a bicharacteristic starting at $(t_0, x_{d0}, x_0', \tau_0, \xi_{d0}, \xi_0')$. The times of collision with the boundary are of the form

$$s_k = \frac{\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) - \frac{(2k+1)\pi}{2}}{\kappa|\xi'|}$$

For $k = -1, 1$ the points of collision have tangential coordinates

$$k = -1 : x_0' + 2\kappa x_{d0}\xi' \left(\frac{|\xi'|^2 + \xi_{d0}^2}{2|\xi'|^2} \frac{\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) - \frac{\pi}{2}}{\kappa|\xi'|} + \frac{\xi_{d0}}{2\kappa|\xi'|^2} \right)$$
$$k = 1 : x_0' + 2\kappa x_{d0}\xi' \left(\frac{|\xi'|^2 + \xi_{d0}^2}{2|\xi'|^2} \frac{\arctan\left(\frac{\xi_{d0}}{|\xi'|}\right) + \frac{\pi}{2}}{\kappa|\xi'|} + \frac{\xi_{d0}}{2\kappa|\xi'|^2} \right)$$

The difference between the times of collision is given by $\Delta s = \dfrac{\pi}{\kappa|\xi'|}$, hence $\Delta t = \dfrac{2\pi\tau_0}{\kappa|\xi'|}$.

The difference in the tangential coordinates is given by $\Delta x' = \pi x_{d0} \dfrac{\xi'}{|\xi'|} \dfrac{|\xi'|^2 + \xi_{d0}^2}{|\xi'|^2}$.

Let $t_0 = 0$, $\tau_0 = 2^j$, $|\xi'| = 2^{2j}$, so that $\Delta t \approx 2^{-j}, \Delta x' \approx 1$. This shows that when $t \in [0, 1]$, the geodesic will hit the boundary 2^j times.

We shall approximate the proportion of time that the geodesic spends in a region of the form $x_d \approx c x_{d0}$, with $c \ll 1$. We recall the expression for $x_d(s)$, and we get that

$$x_d(s) = x_{d0} \left(\frac{1}{2} + \cos\left(2\kappa|\xi'|s\right) \frac{1}{2} \right)$$

We analyze the behavior close to s_1, where $2\kappa|\xi'|s_1 = \pi$. By taking the Taylor series expansion around this point, we get that

$$x_d(s) \approx x_{d0} \left(\frac{1}{2} + \frac{1}{2} \left(-1 + \frac{4\kappa^2|\xi'|^2(s - s_1)^2}{2} \right) \right) \approx x_{d0}\kappa^2|\xi'|^2(s - s_1)^2$$

Thus, the geodesic stays in $x_d \approx c x_{d0}$, with $c \ll 1$ for $\Delta s \approx \dfrac{c^{\frac{1}{2}}}{|\xi'| x_{d0}^{\frac{1}{2}}} \approx 2^{-j} c^{\frac{1}{2}}$, hence for $\Delta t \approx c^{\frac{1}{2}}$.

Therefore, a solution of (2.2.1) which propagates along such a geodesic spends a positive proportion of time in here, bounded from below uniformly in j. In this case, due to the small time variations, the velocity component from the B control parameter would have to be estimated pointwise in time by Sobolev embeddings, the L^1_t component playing thus no role, with the Strichartz estimates not bring any improvement. This suggests that Ifrim and Tataru's result, Theorem 2.2 from [88], might be optimal in the frequency range where $\tau^2 \lesssim |\xi'|$.

2.9 Whispering gallery modes

We consider the operator $\tilde{L} = -\kappa x_d \partial_d^2 - \kappa \Delta_{x'} - \partial_d$. By taking the Fourier transform in the tangential variable x', we obtain $L_{\xi'} = -\kappa x_d \partial_d^2 + \kappa |\xi'|^2 - \partial_d$. We have

$$\begin{aligned} \int L_{\xi'} f \cdot f \, dx_d &= -\int x_d^{\frac{1}{\kappa}-1} (-\kappa x_d |\xi'|^2 f + \kappa x_d \partial_d^2 f + \partial_d f) f \, dx_d \\ &= \kappa \int x_d^{\frac{1}{\kappa}} (\partial_d f)^2 \, dx_d + \int x_d^{\frac{1}{\kappa}-1} f \partial_d f \, dx_d \\ &- \int x_d^{\frac{1}{\kappa}-1} f \partial_d f \, dx_d + \kappa \int x_d^{\frac{1}{\kappa}} |\xi'|^2 f^2 \, dx_d \\ &= \kappa \int x_d^{\frac{1}{\kappa}} (\partial_d f)^2 \, dx_d + \kappa \int x_d^{\frac{1}{\kappa}} |\xi'|^2 f^2 \, dx_d \end{aligned}$$

The associated quadratic form will have the form

$$Q(f) = \int_0^\infty \kappa x_d^{\frac{1}{\kappa}} ((\partial_d f)^2 + |\xi'|^2 f^2) \, dx_d \tag{2.9.1}$$

In what follows, we consider the eigenvalue problem for our operator $\tilde{L}$, which has the form

$$(\kappa x_d \Delta_{x'} + \kappa x_d \partial_d^2 + \partial_d) f = -\lambda f$$

Morally, studying the aforementioned eigenvalue problem corresponds to fixing the time frequency in the wave equation, which is justifiable by the fact that time frequency localization commutes with our equation. Our analysis here will allow us define gallery waves.

For the eigenvalue problem, tangential frequency localization also commutes with $\tilde{L}$, and taking the tangential Fourier transform will give

$$(-\kappa x_d |\xi'|^2 + \kappa x_d \partial_d^2 + \partial_d) f = -\lambda f$$

We also note that $\lambda > 0$, and we write $\lambda = \mu |\xi'|$, where $\mu > 0$, for which our problem now becomes

$$(\mu |\xi'| - \kappa x_d |\xi'|^2 + \kappa x_d \partial_d^2 + \partial_d) f = 0.$$

2.9.1 A WKB analysis for the eigenvalue problem

We distinguish several cases:

The case $\kappa x_d|\xi'|^2 \gtrsim \mu|\xi'|$

Here, we expect the elliptic effect of $x_d\Delta_{x'}$ to dominate. The approximate equation will have the form

$$(-\kappa x_d|\xi'|^2 + \kappa x_d\partial_d^2 + \partial_d)f = 0.$$

We consider an ansatz of the form

$$\tilde{f} = e^{|\xi'|(S_0(x_d)+|\xi'|^{-1}S_1(x_d))},$$

where $|\xi'|$ is our fixed semiclassical parameter.

Inserting this ansatz into the equation, and equating the coefficients of $|\xi'|^2$ and $|\xi'|$ give the solutions

$$\tilde{f} \approx Ke^{\pm|\xi'|x_d}x_d^{-\frac{1}{2\kappa}}$$

However, we seek solutions with the property that $x_d^{\frac{1}{2\kappa}}\partial_d\tilde{f}$ is square integrable, which leaves with the option

$$\tilde{f} \approx Ke^{-|\xi'|x_d}x_d^{-\frac{1}{2\kappa}}.$$

The case $\kappa x_d|\xi'|^2 \ll \mu|\xi'|$, $x_d \gtrsim 1$.

In this case, the uncertainty principle doesn't have a significant effect yet, and we expect the $\mu|\xi'|$ term to dominate, which will impose an oscillatory behavior.

The approximate equation is

$$(\mu|\xi'| + \kappa x_d\partial_d^2 + \partial_d)f = 0.$$

We consider an ansatz of the form

$$\tilde{f} = e^{|\xi'|^{\frac{1}{2}}S_0(x_d)+S_1(x_d)}$$

Inserting this ansatz into the equation, and equating the coefficients of $|\xi'|$ and $|\xi'|^{\frac{1}{2}}$ give the solutions

$$\tilde{f} \approx Kx_d^{\frac{1}{4}-\frac{1}{2\kappa}}e^{\pm|\xi'|^{\frac{1}{2}}\frac{2i\sqrt{\mu}}{\sqrt{\kappa}}x_d^{\frac{1}{2}}},$$

which is square integrable.

The case $\kappa x_d|\xi'|^2 \ll \mu|\xi'|$, $x_d \ll 1$, $\kappa x_d \xi_d^2 \ll |\xi'|\mu$.

We approximately have

$$\partial_d \tilde{f} + |\xi'|\mu \tilde{f} = 0$$
$$\tilde{f} \approx K e^{-x_d|\xi'|\mu}$$

The case $\kappa x_d|\xi'|^2 \ll \mu|\xi'|$, $x_d \ll 1$, $\kappa x_d \xi_d^2 \gtrsim |\xi'|\mu$.

We approximately have

$$\kappa x_d \partial_d^2 \tilde{f} + \partial_d \tilde{f} = 0,$$

hence

$$\tilde{f} = K_1 + K_2 x_d^{1-\frac{1}{\kappa}}$$

when $\kappa \neq 1$.

When $\kappa < 1$, the second solution is not in our desired space, so we would be left with the constant.

When $\kappa = 1$, we would instead have

$$x_d \partial_d^2 \tilde{f} + \partial_d \tilde{f} = 0$$
$$\tilde{f} = K_1 + K_2 \ln(x_d)$$

Here, $\ln(x_d)$ is not in our desired space, so we will be left with the constant.

When $\kappa > 1$, we shall make a choice that is going to be explicitly described in Section 2.9.

For $\xi' \neq 0$, the quadratic form corresponding to $L_{\xi'}$ is non-degenerate and positive definite. In light of our previous discussion, we shall take its domain to be

$$D(Q) = \begin{cases} \{f | x^{\frac{1}{2\kappa}} \partial_x f, x^{\frac{1}{2\kappa}} f, x_d^{\frac{1}{2\kappa}-1} f \in L^2(\mathbb{R}_+)\}, \text{ when } \kappa > 1 \\ \{f | x^{\frac{1}{2\kappa}} \partial_x f, x^{\frac{1}{2\kappa}} f, x_d^{\frac{1}{2\kappa}-1} f \in L^2(\mathbb{R}_+), \lim_{x\to 0_+} x^{\frac{1}{\kappa}-1} f(x) = 0\}, \text{ when } \kappa < 1 \\ \{f | x^{\frac{1}{2\kappa}} \partial_x f, x^{\frac{1}{2\kappa}} f, x_d^{\frac{1}{2\kappa}-1} f \in L^2(\mathbb{R}_+), \lim_{x\to 0_+} \frac{f(x)}{\ln(x)} = 0\}, \text{ when } \kappa = 1. \end{cases}$$

$Q(f)$ is symmetrical, positive definite (bounded from below), so its pointwise spectrum will consist solely of positive real numbers.

By writing $\lambda(\xi') = \mu|\xi'|$, with $\mu_k > 0$, our eigenvalue problem takes the form

$$(\mu|\xi'| - \kappa x_d|\xi'|^2 + \kappa x_d \partial_d^2 + \partial_d) f = 0,$$

where μ is independent of $|\xi'|$ (this can be immediately seen by rescaling).

2.9.2 An explicit form for the solution to the eigenvalue problem

We rewrite the equation solved by the eigenfunction as follows:

$$\left(x_d\partial_d^2 + \frac{1}{\kappa}\partial_d + \left(\frac{\lambda}{\kappa} - |\xi'|^2 x_d\right)\right) f = 0.$$

This equation will have the general solution given by the formula

$$f = c_1 e^{-|\xi'|x_d} U\left(\frac{1}{2}\left(\frac{1}{\kappa} - \frac{\lambda}{\kappa|\xi'|}\right), \frac{1}{\kappa}, 2|\xi'|x_d\right) + c_2 e^{-|\xi'|x_d} L^{\frac{1}{\kappa}-1}_{\frac{1}{2}\left(\frac{\lambda}{\kappa|\xi'|} - \frac{1}{\kappa}\right)}(2|\xi'|x_d)$$

Here, U is a hypergeometric function , which, when b is not an integer, is given by

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\sum_{k=0}^{\infty}\frac{(a)_k z^k}{(b)_k k!} + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b}\sum_{k=0}^{\infty}\frac{(a-b+1)_k z^k}{(2-b)_k k!}\,, b \notin \mathbb{Z},$$

while in the case $b \in \mathbb{Z}_+$,

$$\begin{aligned} U(a,b,z) = {} & \frac{(-1)^b}{\Gamma(a-b+1)}\frac{\log(z)}{(b-1)!}\sum_{k=0}^{\infty}\frac{(a)_k z^k}{(b)_k k!} \\ & + \frac{(-1)^b}{\Gamma(a-b+1)}\frac{\log(z)}{(b-1)!}\sum_{k=0}^{\infty}\frac{(a)_k(\psi(a+k) - \psi(k+1) - \psi(k+b))z^k}{(k+b-1)!k!} \\ & - \frac{(-1)^b}{\Gamma(a-b+1)}\sum_{k=1}^{n-1}\frac{(k-1)!z^{-k}}{(1-a)_k(b-k-1)!}\,, b \in \mathbb{Z}_+. \end{aligned}$$

$L^{\lambda}_{\nu}(z)$ is a generalized Laguerre polynomial given by

$$L^{\lambda}_{\nu}(z) = \frac{\Gamma(\lambda+\nu+1)}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{(-\nu)_k z^k}{\Gamma(k+\lambda+1)k!},$$

and $(a)_k = \Pi_{j=1}^k(a+j-1)$ is the Pochammer symbol.

The discussion from the previous section allows us to choose

$$f(x_d) = e^{-|\xi'|x_d} L^{\frac{1}{\kappa}-1}_{\frac{1}{2}\left(\frac{\lambda}{\kappa|\xi'|} - \frac{1}{\kappa}\right)}(2|\xi'|x_d).$$

When $\kappa \leq 1$, an equivalent way to impose this choice is to require that
$$\begin{cases} \lim_{x\to 0_+} \frac{f(x)}{\ln(x)} = 0, \kappa = 1 \\ \lim_{x\to 0_+} x^{\frac{1}{\kappa}-1} f(x) = 0, \kappa < 1. \end{cases}$$

In all cases, we also know that f is bounded, and that its zeroes are isolated (it can be extended to a holomorphic function in the whole complex plane). Moreover, it will also follow that f is continuous with respect to the parameters $|\xi'|$, λ, and κ.

Definition 2.10. Let $\mu > 0$, $\xi' \neq 0$ be a non-zero tangential frequency, the corresponding eigenvalue $\lambda = \mu|\xi'|$, and let $B(\mu, x) := f\left(\frac{x}{|\xi'|}\right)$, where f is as above. We define the set of whispering gallery modes $E_\mu(\Omega)$ by taking the closure in $\{f | x^{\frac{1}{2\kappa}-1} f \in L^2(\mathbb{R}_+ \times \mathbb{R}^{d-1})\}$ of

$$\left\{ u(x_d, x') = \frac{1}{(2\pi)^{d-1}} \int e^{ix'\cdot\xi'} B(\mu, |\xi'|x_d)\hat{\varphi}(\xi')\, d\xi', \varphi \in \mathscr{S}(\mathbb{R}^{d-1}) \right\}.$$

In particular, $u \in E_\mu(\Omega)$ is called a *whispering gallery mode.*

We note that every $u \in E_\mu(\Omega)$ solves

$$(\kappa x_d \Delta + \partial_d + \mu|\nabla_{x'}|)u = 0.$$

We shall also need the following equivalence between the norms of $\varphi \in \mathscr{S}(\mathbb{R}^{d-1})$, and its associated element $u_0 \in E_\mu(\Omega)$:

Lemma 2.11. Let $\varphi \in \mathscr{S}(\mathbb{R}^{d-1})$, and u_0 be defined by

$$u(x_d, x') = \frac{1}{(2\pi)^{d-1}} \int e^{ix'\cdot\xi'} B(\mu, |\xi'|x)\hat{\varphi}(\xi')\psi\left(\frac{\xi'}{2^{2j}}\right) d\xi'$$

Then, if $\psi, \psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^{d-1})$ such that $\psi\psi_1 = \psi_1$, and $\psi\psi_2 = \psi$, there exist constants $C_1, C_2 > 0$ for which

$$C_1 \left\| \psi_1\left(\frac{D_{x'}}{2^{2j}}\right)\varphi \right\|_{L^r_{x'}(\mathbb{R}^{d-1})} \leq (2^{2j})^{-\frac{1}{r}} \left\| \psi\left(\frac{D_{x'}}{2^{2j}}\right) u\left(\frac{s}{2^{2j}}, \cdot\right) \right\|_{L^r_x(\mathbb{R}_+\times\mathbb{R}^{d-1})}$$

and

$$(2^{2j})^{-\frac{1}{r}} \left\| \psi\left(\frac{D_{x'}}{2^{2j}}\right) u\left(\frac{s}{2^{2j}}, \cdot\right) \right\|_{L^r_x(\mathbb{R}_+\times\mathbb{R}^{d-1})} \leq C_2 \left\| \psi_2\left(\frac{D_{x'}}{2^{2j}}\right)\varphi \right\|_{L^r_{x'}(\mathbb{R}^{d-1})}$$

Moreoever,

$$\left\| x_d^{\frac{1}{2\kappa}} \psi\left(\frac{D_{x'}}{2^{2j}}\right) u \right\|_{L^2_x(\mathbb{R}_+\times\mathbb{R}^{d-1})} \approx (2^{2j})^{-\frac{1}{2\kappa}-\frac{1}{2}} \left\| \psi\left(\frac{D_{x'}}{2^{2j}}\right)\varphi \right\|_{L^2_{x'}(\mathbb{R}^{d-1})}$$

$$\left\| x_d^{\frac{1}{2\kappa}} \psi\left(\frac{D_{x'}}{2^{2j}}\right) \nabla_x u \right\|_{L^2_x(\mathbb{R}_+\times\mathbb{R}^{d-1})} \approx (2^{2j})^{-\frac{1}{2\kappa}+\frac{1}{2}} \left\| \psi\left(\frac{D_{x'}}{2^{2j}}\right)\varphi \right\|_{L^2_{x'}(\mathbb{R}^{d-1})}$$

Proof. We have

$$\begin{aligned}
\psi(2^{-2j}D_{x'})u(x_d, x') &= \frac{1}{(2\pi)^{d-1}} \int e^{ix'\cdot\xi'} B(\mu, |\xi'|x_d)\hat{\varphi}(\xi')\psi\left(\frac{\xi'}{2^{2j}}\right) d\xi' \\
&= \frac{2^{2j(d-1)}}{(2\pi)^{d-1}} \int e^{i2^{2j}x'\cdot\eta} B(\mu, 2^{2j}|\eta|x_d)\hat{\varphi}(2^{2j}\eta)\psi(\eta)\, d\eta
\end{aligned}$$

By making the change of variables $x_d = s2^{-2j}$, we reduce the inequality to proving that

$$C_1 \left\| \psi_1 \left(\frac{D_{x'}}{2^{2j}} \right) \varphi \right\|_{L^r_{x'}(\mathbb{R}^{d-1})} \leq \left\| \psi \left(\frac{D_{x'}}{2^{2j}} \right) u(2^{-2j}s, \cdot) \right\|_{L^r_x(\mathbb{R}_+ \times \mathbb{R}^{d-1})} \leq C_2 \left\| \psi_2 \left(\frac{D_{x'}}{2^{2j}} \right) \varphi \right\|_{L^r_{x'}(\mathbb{R}^{d-1})}$$

We may assume that $\operatorname{supp} \psi \subset \{\eta \in \mathbb{R}^{d-1} | 0 < c_0 \leq |\eta| \leq c_1\}$.

We also have

$$\psi(2^{-2j} D_{x'}) u(2^{2j}s, x') = \frac{2^{2j(d-1)}}{(2\pi)^{d-1}} \int e^{i2^{2j}x' \cdot \eta} B(\mu, |\eta| s) \hat{\varphi}(2^{2j}\eta) \psi(\eta)\, d\eta$$

We know that $B(\mu, 0) > 0$. Thus, there exists an interval $[a, b]$ such that if $s \in [a, b]$, $B(\mu, |\eta| s) \geq \delta > 0$ for some constant δ, and every $\eta \in \operatorname{supp} \psi$, hence here $B(\mu, |\eta| s)$ is positive and bounded. As in [90], it follows that

$$\|\psi_1(2^{-2j} D_{x'}) \psi(2^{-2j} D_{x'}) \varphi\|_{L^r_{x'}(\mathbb{R}^{d-1})} \leq C_1 \|\psi(2^{-2j} D_{x'}) \varphi\|_{L^r_s L^r_{x'}([a,b] \times \mathbb{R}^{d-1})},$$

where

$$C_1 = \frac{\sup_\eta |\psi_1(\eta)|}{\inf_{c \in [a,b]} |B(\mu, c)|},$$

which implies that

$$\|\psi_1(2^{-2j} D_{x'}) \varphi\|_{L^r_{x'}(\mathbb{R}^{d-1})} \leq C_1 \|\psi(2^{-2j} D_{x'}) \varphi\|_{L^r_s L^r_{x'}(\mathbb{R}_+ \times \mathbb{R}^{d-1})}.$$

The second inequality is immediate, as $\psi = \psi\psi_2$, along with the fact that $|B(\mu, c)|)$ is bounded immediately imply that

$$\|\psi(2^{-2j} D_{x'}) \varphi\|_{L^r_s L^r_{x'}(\mathbb{R}_+ \times \mathbb{R}^{d-1})} \leq C_2 \|\psi_2(2^{-2j} D_{x'}) \psi(2^{-2j} D_{x'}) \varphi\|_{L^r_{x'}(\mathbb{R}^{d-1})},$$

where

$$C_2 = \sup_\eta |\psi(\eta)| \sup_c |B(\mu, c)|$$

For the second estimate, Plancherel's Theorem implies that

$$\begin{aligned}
\left\| x_d^{\frac{1}{2\kappa}} \psi \left(\frac{D_{x'}}{2^{2j}} \right) u \right\|^2_{L^2_x(\mathbb{R}_+ \times \mathbb{R}^{d-1})} &\approx \int_0^\infty \int_{\mathbb{R}^{d-1}} x_d^{\frac{1}{\kappa}} \left| B(\mu, |\xi'| x_d) \hat{\varphi}(\xi') \psi \left(\frac{\xi'}{2^{2j}} \right) \right|^2 d\xi'\, dx_d \\
&= \int_{\mathbb{R}^{d-1}} \int_0^\infty x_d^{\frac{1}{\kappa}} (B(\mu, |\xi'| x_d))^2\, dx_d \left| \psi \left(\frac{\xi'}{2^{2j}} \right) \hat{\varphi}(\xi') \right|^2 d\xi' \\
&= \int_{\mathbb{R}^{d-1}} |\xi'|^{-\frac{1}{\kappa}} \int_0^\infty (x_d |\xi'|)^{\frac{1}{\kappa}} (B(\mu, |\xi'| x_d))^2\, dx_d \left| \psi \left(\frac{\xi'}{2^{2j}} \right) \hat{\varphi}(\xi') \right|^2 d\xi' \\
&= \int_{\mathbb{R}^{d-1}} |\xi'|^{-\frac{1}{\kappa} - 1} \int_0^\infty y_d^{\frac{1}{\kappa} - 1} (B(\mu, y_d))^2\, dy_d \left| \psi \left(\frac{\xi'}{2^{2j}} \right) \hat{\varphi}(\xi') \right|^2 d\xi' \\
&\approx (2^{2j})^{-\frac{1}{\kappa} - 1} \left\| \psi \left(\frac{D_{x'}}{2^{2j}} \right) \varphi \right\|^2_{L^2_{x'}(\mathbb{R}^{d-1})}
\end{aligned}$$

Therefore,

$$\left\| x_d^{\frac{1}{2\kappa}} \psi\left(\frac{D_{x'}}{2^{2j}}\right) u \right\|_{L_x^2(\mathbb{R}_+\times\mathbb{R}^{d-1})} \approx (2^{2j})^{-\frac{1}{2\kappa}-\frac{1}{2}} \left\| \psi\left(\frac{D_{x'}}{2^{2j}}\right) \varphi \right\|_{L_{x'}^2(\mathbb{R}^{d-1})}$$

The last estimate is analogous. □

Remark 2.11.1. Let $\varphi_0 \in \mathscr{S}(\mathbb{R}^{d-1})$, and $u_0 \in E_\mu(\Omega)$ be such that

$$u_0(x_d, x') = \frac{1}{(2\pi)^{d-1}} \int e^{ix'\cdot\xi'} B(\mu, |\xi'|x_d)\hat{\varphi}(\xi')\, d\xi'.$$

In order to prove Theorem 2.5.2, we may reduce the problem to the study of Strichartz type estimates for a problem with initial data φ_0. More precisely, if u solves

$$\begin{aligned}(\partial_t^2 - \kappa x_d \Delta - \partial_d)u = 0\\ (u, \partial_t u)(0) = (P_{x',2^{2j}} u_0, 0)\end{aligned}$$

then in order to show that the gallery modes satisfy the estimates from Theorem 2.5.2, it suffices to show that the solution to

$$\begin{aligned}(\partial_t^2 + \mu|\nabla_{x'}|)\varphi = 0\\ (\varphi, \partial_t\varphi)(0) = (P_{x',2^{2j}}\varphi_0, 0)\end{aligned}$$

satisfies

$$\|f\|_{L_t^q L_{x'}^r([0,T]\times\mathbb{R}^{d-1})} \lesssim 2^{2j\left(\frac{3(d-1)}{4}\right)\left(\frac{1}{2}-\frac{1}{r}\right)} \|P_{x',2^{2j}} f_0\|_{L_{x'}^2(\mathbb{R}^{d-1})},$$

Proof. Let φ be the solution of the initial value problem

$$\begin{aligned}(\partial_t^2 + \mu|\nabla_{x'}|)\varphi = 0\\ (\varphi, \partial_t\varphi)(0) = (P_{x',2^{2j}}\varphi_0, 0)\end{aligned}$$

We define

$$u(t, x) = \frac{1}{(2\pi)^{d-1}} \int e^{ix'\cdot\xi'} B(\mu, |\xi'|x_d)\hat{\varphi}(t, \xi')\, d\xi'.$$

Here, $\hat{\varphi}$ is the Fourier transform with respect to the tangential variables.

We claim that u is the solution to the problem

$$\begin{aligned}(\partial_t^2 - \kappa x_d \Delta - \partial_d)u = 0\\ (u, \partial_t u)(0) = (P_{x',2^{2j}} u_0, 0).\end{aligned}$$

Indeed, by using that $(\kappa x_d\partial_d^2+\partial_d)B(\mu,|\xi'|x_d)=(\kappa x_d|\xi'|^2-\mu|\xi'|)B(\mu,|\xi'|x_d)$, we find that

$$(\partial_t^2-\kappa x_d\Delta_{x'}-\kappa x_d\partial_d^2-\partial_d)u=\frac{1}{(2\pi)^{d-1}}\int e^{ix'\cdot\xi'}B(\mu,|\xi'|x_d)(\partial_t^2\hat{\varphi}(t,\xi')+\mu|\xi'|\hat{\varphi})\,d\xi'=0.$$

Here, we have used the fact that

$$(\partial_t^2+\mu|\nabla_{x'}|)\varphi=0,$$

which implies that

$$\partial_t^2\hat{\varphi}(t,\xi')+\mu|\xi'|\hat{\varphi}=0.$$

The initial data conditions are also clearly true.

The rest of the proof immediately follows from Lemma 2.11. □

2.12 Proof of Theorem 2.5.2

From Corollary 2.11.1, we know that it suffices to prove the following

Proposition 2.13. If f is a solution of

$$\begin{aligned}(\partial_t^2+\mu|\nabla_{x'}|)f&=0\\(f,\partial_t f)(0)&=(P_{x',2^{2j}}f_0,0),\end{aligned}$$

then

$$\|f\|_{L_t^qL_{x'}^r}\lesssim\|P_{x',2^{2j}}f_0\|_{L_{x'}^2}.$$

Proof. Let us first take $f_0=\varphi_0\in\mathscr{S}(\mathbb{R}^{d-1})$. The solution to our problem has the form

$$f=\frac{1}{(2\pi)^{d-1}}\int e^{ix'\cdot\xi'}e^{-it\mu^{\frac12}|\xi'|^{\frac12}}\psi\left(\frac{\xi'}{2^{2j}}\right)\hat{f}_0(\xi')\,d\xi'$$

This can be rewritten as

$$\begin{aligned}f&=\frac{1}{(2\pi)^{d-1}}\int e^{ix'\cdot\xi'}e^{-it\mu^{\frac12}|\xi'|^{\frac12}}\psi\left(\frac{\xi'}{2^{2j}}\right)\hat{f}_0(\xi')\,d\xi'\\&=\frac{2^{2j(d-1)}}{(2\pi)^{d-1}}\int e^{i2^{2j}x'\cdot\eta}e^{-it2^j\mu^{\frac12}|\eta|^{\frac12}}\psi(\eta)\,\hat{f}_0(2^{2j}\eta)\,d\eta\\&=\frac{2^{2j(d-1)}}{(2\pi)^{d-1}}\int e^{it2^{2j}\left(\frac{x'}{t}\cdot\eta-2^{-j}\mu^{\frac12}|\eta|^{\frac12}\right)}\psi(\eta)\,\hat{f}_0(2^{2j}\eta)\,d\eta\end{aligned}$$

Let

$$G(j,\eta) = 2^{-j}\mu^{\frac{1}{2}}|\eta|^{\frac{1}{2}}\,, \lambda = t2^{2j}\,, z = \frac{x'}{t}$$

$$J(z,j,\lambda) := \int e^{i\lambda(z\cdot\eta - G(j,\eta))}\psi(\eta)\, d\eta$$

$$\gamma_{d-1}(j,\lambda) = \sup_{z\in\mathbb{R}^{d-1}} |J(z,j,\lambda)|$$

When λ is small, the bounds follow immediately. For large λ, we are going to apply the following

Lemma 2.14 (Lemma 3.3,[134])**.** Let a be a bump function, and let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth and have a stationary point at x_0 with $\det(\nabla^2\phi(x_0)) \neq 0$. If ϕ has no other stationary points on the support of a, then there exist constants c_0, c_1,... with each c_n depending (in some explicit fashion) only on finitely many derivatives of a, ϕ at x_0, such that we have the asymptotic formula

$$\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}a(x)\, dx = \sum_{n=0}^{N} c_n\lambda^{-n-\frac{d}{2}} + O_{a,\phi,\lambda}(\lambda^{-N-\frac{d}{2}-1}),$$

for all $N \geq 0$. Furthermore,

$$c_0 = e^{\frac{i\pi\,\mathrm{sgn}(\nabla^2\phi(x_0))}{4}} e^{i\lambda\phi(x_0)} \sqrt{\frac{2\pi}{|\det\nabla^2\phi(x_0)|}}a(x_0).$$

The critical point of the phase is given by

$$z = 2^{-j}\mu^{\frac{1}{2}}\frac{\eta}{2|\eta|^{\frac{3}{2}}},$$

while the Hessian of G at the critical point is

$$(\nabla^2 G(j,\eta))_{ij} = \frac{2^{-j}\mu^{\frac{1}{2}}}{2|\eta|^{\frac{3}{2}}}\left(\delta_{ij} - \frac{3\eta_i\eta_j}{2|\eta|^2}\right),$$

which is nondegenerate. Lemma 2.14 implies that

$$J(z,j,\lambda) \approx \left(\frac{2\pi}{\lambda^{\frac{1}{2}}}\right)^{d-1} \frac{e^{-\frac{i\pi}{4}\,\mathrm{sgn}\,\nabla^2 G(\eta(z))}}{\sqrt{|\det\nabla^2 G(\eta(z))|}}\psi(\eta(z)),$$

hence

$$|J(z,j,\lambda)| \approx (t2^{2j})^{-\frac{d-1}{2}} 2^{\frac{j(d-1)}{2}} |\eta(z)|^{\frac{3(d-1)}{4}},$$

and as η is in suppψ (which is a neighbourhood of a the unit sphere) and away from 0,

$$|J(z,j,\lambda)| \lesssim t^{-\frac{d-1}{2}} 2^{-j\left(\frac{d-1}{2}\right)}$$

This immediately implies that we may take

$$\gamma_{d-1}(j,\lambda) := t^{-\frac{d-1}{2}} 2^{-j\left(\frac{d-1}{2}\right)}$$

We are now going to use the following

Lemma 2.15. [Lemma 2.2,[90]] Let $\alpha \geq$ and (q,r) be such that $\dfrac{1}{q} + \dfrac{\alpha}{r} = \dfrac{\alpha}{2}$, with $q > 2$, in dimension n. Let $\beta = (n-\alpha)\left(\dfrac{1}{2} - \dfrac{1}{r}\right)$. If the solution $u = e^{-\frac{it}{h}G}\psi(hD)u_0$ of the initial value problem

$$\frac{h}{i}\partial_t u - G\left(\frac{hD}{i}\right)u = 0$$
$$u_{|t=0} = \psi(hD)u_0$$

satisfies the dispersive estimates

$$\left\| e^{-\frac{it}{h}G}\psi(hD)u_0 \right\|_{L^\infty_x(\mathbb{R}^n)} \lesssim \frac{1}{(2\pi h)^n}\gamma_{n,h}\left(\frac{t}{h}\right)\|\psi(hD)u_0\|_{L^1_x(\mathbb{R}^n)}$$

for some function $\gamma_{n,h} : \mathbb{R} \to [0,\infty)$, and every $t \in (0, T_0]$, then there exists some $C > 0$ independent of h, such that the following inequality holds

$$h^\beta \left\| e^{-\frac{it}{h}G}\psi(hD)u_0 \right\|_{L^q_t L^r_x((0,T_0]\times\mathbb{R}^n)} \leq C \left(\sup_{s\in\left(0,\frac{T_0}{h}\right)} s^\alpha \gamma_{n,h}(s) \right)^{\frac{1}{2}-\frac{1}{r}} \|u_0\|_{L^2_x(\mathbb{R}^n)}$$

$$\begin{aligned} \|f(t)\|_{L^\infty_{x'}(\mathbb{R}^{d-1})} &\lesssim 2^{2j(d-1)}\gamma_{d-1}(j,\lambda)\|f_0\|_{L^1_{x'}(\mathbb{R}^{d-1})} \\ &\lesssim t^{-\frac{d-1}{2}} 2^{j\left(\frac{3(d-1)}{2}\right)}\|f_0\|_{L^1_{x'}(\mathbb{R}^{d-1})} \end{aligned}$$

By interpolation,

$$\|f(t)\|_{L^r_{x'}(\mathbb{R}^{d-1})} \lesssim t^{-\frac{d-1}{2}\left(1-\frac{2}{r}\right)} 2^{j\left(\frac{3(d-1)}{2}\right)\left(1-\frac{2}{r}\right)}\|f_0\|_{L^{r'}_{x'}(\mathbb{R}^{d-1})}$$

Lemma 2.15 now implies that

$$\|f\|_{L^q_t L^r_{x'}([0,T]\times\mathbb{R}^{d-1})} \lesssim 2^{2j\left(\frac{3(d-1)}{4}\right)\left(\frac{1}{2}-\frac{1}{r}\right)}\|P_{x',2^{2j}}f_0\|_{L^2_{x'}(\mathbb{R}^{d-1})},$$

which finishes the proof. □

Now, Lemma 2.11, implies that

$$\|f\|_{L^q_t L^r_x([0,T]\times((\mathbb{R}_+\times\mathbb{R}^{d-1})))} \lesssim (2^{2j})^{\left(\frac{3d+1}{4}\right)\left(\frac{1}{2}-\frac{1}{r}\right)+\frac{1}{2\kappa}-1}\|(0,\nabla_x P_{x',2^{2j}}f_0)\|_{\mathscr{H}},$$

which finishes the proof of Theorem 2.5.2.

2.16 Construction of our counterexamples to the Strichartz estimates

In this section, we construct the functions that are going to serve as our counterexamples to the classical Strichartz for the wave equation.

Let $a \in C_0^\infty(\mathbb{R}^d)$ be an even function with

$$\operatorname{supp} a \in \left\{ \tilde{\xi'} | 1 - \varepsilon(\kappa) \leq |\tilde{\xi'}| \leq 1 + \varepsilon(\kappa) \right\}.$$

For $j \geq 0$, we define

$$U^j(t, x_d, x') = \frac{1}{(2\pi)^{d-1}} e^{it2^j} \int e^{ix' \cdot \xi'} B\left(\frac{2^{2j}}{|\xi'|}, |\xi'| x_d\right) a(2^{-2j}\xi')\, d\xi'$$

We note that

$$\begin{aligned} U^j(t, x_d, x') &= \frac{1}{(2\pi)^{d-1}} e^{it2^j} \int e^{ix' \cdot \xi'} B\left(\frac{2^{2j}}{|\xi'|}, |\xi'| x_d\right) a(2^{-2j}\xi')\, d\xi' \\ &= \frac{2^{2j(d-1)}}{(2\pi)^{d-1}} e^{it2^j} \int e^{i2^{2j} x' \cdot \eta} B\left(\frac{1}{|\eta|}, |\eta| 2^{2j} x_d\right) a(\eta)\, d\eta \\ &= 2^{2j(d-1)} U^0(2^j t, 2^{2j} x) \end{aligned}$$

It is clear that U^j is an exact solution to our wave equation

$$(\partial_t^2 - \kappa x_d \Delta_{x'} - \kappa x_d \partial_d^2 - \partial_d) U^j = 0$$

Proposition 2.17. Let U^j be defined as above. Then, for $t \in [0, 1]$, we have $\|U^j(t, \cdot)\|_{L^r_x(\Omega)} \simeq (2^{2j})^{d-1-\frac{d}{r}}$, uniformly in t. Moreover, $\|U^j\|_{\mathscr{H}} \approx 2^{2j\left(\frac{d}{2} - \frac{1}{\kappa}\right)}$.

Proof. From Lemma 2.11, we immediately have

$$\begin{aligned} \|U^j(t, x_d)\|_{L^r_x((0,\infty)\times\mathbb{R}^{d-1}))} &\approx (2^{2j})^{-\frac{1}{r}} 2^{2j(d-1)} \|\tilde{a}(2^{2j} x')\|_{L^r_{x'}(\mathbb{R}^{d-1})} \\ &\approx (2^{2j})^{-\frac{1}{r}} (2^{2j})^{d-1-\frac{d-1}{r}} \approx (2^{2j})^{d-1-\frac{d}{r}} \end{aligned}$$

We also estimate the $\mathscr{H}$-norm of the initial data.

$$\begin{aligned} U^j(0) &= \frac{1}{(2\pi)^{d-1}} \int e^{ix' \cdot \xi'} B\left(\frac{2^{2j}}{|\xi'|}, |\xi'| x_d\right) a(2^{-2j}\xi')\, d\xi' \\ \partial_t U^j(0) &= 2^j \frac{1}{(2\pi)^{d-1}} \int e^{ix' \cdot \xi'} B\left(\frac{2^{2j}}{|\xi'|}, |\xi'| x_d\right) a(2^{-2j}\xi')\, d\xi' = 2^j U(0, x_d, x') \end{aligned}$$

We have

$$\begin{aligned}
\|x_d^{\frac{1}{2\kappa}-\frac{1}{2}}\partial_t U^j(0,x_d,x')\|_{L^2_x(\Omega)} &= 2^{2j(d-1)}2^j\|x_d^{\frac{1}{2\kappa}-\frac{1}{2}}U^0(0,2^{2j}x_d,2^{2j}x')\|_{L^2_x(\Omega)} \\
&= 2^j 2^{2j(d-1)}2^{2j\left(\frac{1}{2}-\frac{1}{2\kappa}\right)}\|(2^{2j}x_d)^{\frac{1}{2\kappa}-\frac{1}{2}}U^0(0,2^{2j}x_d,2^{2j}x')\|_{L^2_x(\Omega)} \\
&= 2^j 2^{2j(d-1)}2^{2j\left(\frac{1}{2}-\frac{1}{2\kappa}\right)}2^{-jd}\|(y_d)^{\frac{1}{\kappa}-\frac{1}{2}}U^0(0,y_d,y')\|_{L^2_y(\Omega)} \\
&\approx 2^{2j\left(\frac{d}{2}-\frac{1}{2\kappa}\right)}
\end{aligned}$$

Similarly,

$$\|x_d^{\frac{1}{2\kappa}}\nabla_x U^j(0,x_d,x')\|_{L^2_x(\Omega)} \approx 2^{2j\left(\frac{d}{2}-\frac{1}{2\kappa}\right)},$$

hence

$$\|U^j(0)\|_{\mathscr{H}} \approx 2^{2j\left(\frac{d}{2}-\frac{1}{2\kappa}\right)}$$

This finishes the proof.

□

2.18 Proof of Theorems 2.4.1 and 2.5.1

We define

$$\psi^j = \frac{U^j}{2^{2j\left(\frac{d}{2}-\frac{1}{2\kappa}\right)}};$$

For a wave-admissible pair (q,r), we have

$$\|\psi^j(t,\cdot)\|_{L^q_tL^r_x([0,1]\times\Omega)} \approx 2^{2j\left(d-1-\frac{d}{r}+\frac{1}{2\kappa}-\frac{d}{2}\right)} \approx 2^{2j\left(\frac{1}{q}+\gamma+\frac{1}{2\kappa}-1\right)},$$

hence

$$\|\nabla_x^2\psi^j(t,\cdot)\|_{L^q_tL^r_x([0,1]\times\Omega)} \approx 2^{2j\left(d+1-\frac{d}{r}+\frac{1}{2\kappa}-\frac{d}{2}\right)} \approx 2^{2j\left(\frac{1}{q}+\gamma+\frac{1}{2\kappa}+1\right)}.$$

It is also clear that

$$\|(\partial_t\psi_0^j,\nabla_x\psi_0^j)\|_{\mathscr{H}} \approx 1,$$

and Bernstein's inequality also implies that

$$\|(\partial_t\psi_0^j,\nabla_x\psi_0^j)\|_{\mathscr{H}^{2s}} \approx 2^{2js}.$$

Therefore, whenever $\alpha \in \left(0, \dfrac{1}{q}+\gamma+\dfrac{1}{2\kappa}+1-s\right)$, we have

$$2^{-2j\alpha}\|\nabla^2\psi^j(t,\cdot)\|_{L^q_tL^r_x([0,1]\times\Omega)} \gg \|(\partial_t\psi_0^j,\nabla_x\psi_0^j)\|_{\mathscr{H}^{2s}},$$

which finishes the proof of Theorem 2.4.1.

When (q,r) is instead Euler-admissible, we have

$$\begin{aligned}\|\psi^j(t,\cdot)\|_{L^q_t L^r_x([0,1]\times\Omega)} &\approx 2^{2j\left(d-1-\frac{d}{r}+\frac{1}{2\kappa}-\frac{d}{2}\right)} \approx 2^{2j\left(\frac{1}{2q}-\gamma\right)}\\ \|\nabla^2\psi^j(t,\cdot)\|_{L^q_t L^r_x([0,1]\times\Omega)} &\approx 2^{2j\left(\frac{1}{2q}-\gamma+2\right)},\end{aligned}$$

hence whenever $\alpha \in \left(0, \dfrac{1}{2q} - \gamma + 2 - s\right)$, we have

$$2^{-2j\alpha}\|\nabla^2\psi^j(t,\cdot)\|_{L^q_t L^r_x([0,1]\times\Omega)} \gg \|(\partial_t\psi^j_0, \nabla_x\psi^j_0)\|_{\mathscr{H}^{2s}},$$

We also note that, when $d \geq 3$, and $(q,r) = (2,\infty)$, we need at least $s > k_0 + \dfrac{1}{2}$ (hence $2s > 2k_0 + 1$) and this suggests that Strichartz estimates can not be used in order to control $\|\nabla^2_x\psi^j\|_{L^2_t L^\infty_x([0,T]\times\Omega)}$ and improve Ifrim and Tataru's local well-posedness result from [88].

Chapter 3

The surface quasi-geostrophic front equation

3.1 Introduction

This chapter is going to be concerned with the results of the preprint [3]. The surface quasi-geostrophic (SQG) equation takes the form

$$\theta_t + u \cdot \nabla\theta = 0, \qquad u = (-\Delta)^{-\frac{1}{2}}\nabla^{\perp}\theta \tag{3.1.1}$$

where θ is a scalar evolution on $\mathbb{R}^2$, $(-\Delta)^{-\frac{1}{2}}$ denotes a fractional Laplacian, and $\nabla^{\perp} = (-\partial_y, \partial_x)$. The SQG equation arises from oceanic and atmospheric science as a model for quasi-geostrophic flows confined to a surface. This equation is also of interest due to similarities with the three dimensional incompressible Euler equation. In particular, the question of singularity formation remains open for both problems.

The SQG equation is one member in a family of two-dimensional active scalar equations parameterized by the transport term in (3.1.1), with

$$u = (-\Delta)^{-1+\frac{\alpha}{2}}\nabla^{\perp}\theta, \qquad \alpha \in [0, 2). \tag{3.1.2}$$

The case $\alpha = 0$ gives the two dimensional incompressible Euler equation, while the $\alpha = 1$ case gives the SQG equation (3.1.1) above. For further analysis of the SQG equation, see Resnick [120].

Front solutions to (3.1.1) refer to piecewise constant solutions taking the form

$$\theta(t, x, y) = \begin{cases} \theta_+ & \text{if } y > \varphi(t, x), \\ \theta_- & \text{if } y < \varphi(t, x), \end{cases}$$

where the front is modeled by the graph $y = \varphi(t, x)$ with $x \in \mathbb{R}$. Front solutions are closely related to patch solutions

$$\theta(t, x, y) = \begin{cases} \theta_+ & \text{if } (x, y) \in \Omega(t), \\ \theta_- & \text{if } (x, y) \notin \Omega(t), \end{cases}$$

where Ω is a bounded, simply connected domain. For instance, when $\alpha \in (1, 2)$, the derivation and analysis of contour dynamics equations do not differ substantially between the case of patches and the case of fronts. Global well-posedness for the front equation for small and localized data was established by Córdoba-Gómez-Serrano-Ionescu in [39].

However, when $\alpha \in [0, 1]$, the derivation of contour dynamics equations for fronts has additional complexities relative to the case of patches, arising from the slow decay of Green's functions. The derivation in this range was provided by Hunter-Shu [74] via a regularization procedure, and again by Hunter-Shu-Zhang in [76]. In the SQG case $\alpha = 1$, the evolution equation for the front φ takes the form

$$\begin{aligned} (\partial_t - 2\log|D_x|\partial_x)\varphi(t, x) &= Q(\varphi, \partial_x\varphi)(t, x), \\ \varphi(0, x) &= \varphi_0(x), \end{aligned} \tag{3.1.3}$$

where φ is a real-valued function $\varphi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and

$$Q(f, g)(x) = \int \left(\frac{1}{|y|} - \frac{1}{\sqrt{y^2 + (f(x+y) - f(x))^2}} \right) \cdot (g(x+y) - g(x))\, dy. \tag{3.1.4}$$

This can be rewritten using difference quotients as

$$Q(f, g)(x) = \int F(\delta^y f) \cdot |\delta|^y g\, dy, \tag{3.1.5}$$

where

$$F(s) = 1 - \frac{1}{\sqrt{1+s^2}}, \qquad \delta^y f(x) = \frac{f(x+y) - f(x)}{y}, \qquad |\delta|^y g(x) = \frac{g(x+y) - g(x)}{|y|}.$$

The equation (3.1.3) is invariant under the transformation

$$t \to \kappa t, \qquad x \to \kappa(x + 2t\log|\kappa|), \qquad \varphi \to \kappa\varphi,$$

which implies that $\dot{H}^{\frac{3}{2}}(\mathbb{R})$ is the corresponding critical Sobolev space.

In the case of SQG patches, the first local well-posedness results were obtained by Rodrigo [121] for initial data in C^∞. Later on, Gancedo-Nguyen-Patel proved in [55] that under a suitable parametrization, the contour dynamics evolution is locally well-posed in $H^s(\mathbb{T})$ when $s > 2$. For the generalized SQG family, the first local well-posedness results were obtained by Gancedo [54] in the case $\alpha \in (0,1]$, for which they showed that the problem is locally well-posed in $H^3(\mathbb{T})$. Later on, Gancedo-Patel analyzed the gSQG case with $\alpha \in (0,2)$ and $\alpha \neq 1$ in [56], where they in particular obtained local well-posedness in H^2 for $\alpha \in (0,1)$ and in H^3 for $\alpha \in (1,2)$. For a more recent result on enhanced lifespan for α-patches, see Berti-Cuccagna-Gancedo-Scrobogna [20].

Concerning ill-posedness for patches, Kiselev-Luo [104] proved results in Sobolev spaces with exponents $p \neq 2$, as well as in Hölder spaces. Further, Zlato**v**s [148] proved finite time blow up for patch solutions (both bounded and unbounded) for suitable initial data and $\alpha \in (0,1/4]$, provided that local well-posedness for (3.1.1) is known in the Sobolev space $H^3(\mathbb{R}^2)$ (corresponding to solutions in $H^2(\mathbb{R})$ in the contour dynamics formulation from [56]).

In the current article we are concerned with SQG fronts. Work in this area began with Hunter-Shu-Zhang, who studied the local well-posedness for a cubic approximation of (3.1.3) in [75], and subsequently in [77] established local well-posedness for the full equation (3.1.3) with initial data in H^s, $s \geq 5$, along with global well-posedness for small, localized, and essentially smooth ($s \geq 1200$) initial data. These results were extended to the range $\alpha \in (1,2)$ for the generalized SQG family in [78], while also improving the required regularity to $s > \frac{7}{2} + \frac{3\alpha}{2}$. However, these well-posedness results require a small data assumption to ensure the coercivity of the modified energies used in the energy estimates, along with a convergence condition on an expansion of the nonlinearity $Q(\varphi, \varphi_x)$ appearing in (3.1.3).

In [6], the authors lowered the regularity thresholds for both the local and global theory, proving that the problem is locally well-posed in H^s when $s > \frac{5}{2}$, which corresponds to the classical energy threshold of Hughes-Kato-Marsden [69] at one derivative above scaling; and globally well-posed for small and localized initial data in H^s when $s > 4$.

In the present article, our objective is to revisit and streamline the analysis of (3.1.3), while improving the established well-posedness results. Our contributions include:

- establishing the local well-posedness in a significantly lower regularity setting at $\frac{1}{2} + \epsilon$ derivatives above scaling, by observing a null structure for the equation and carrying out an associated normal form analysis, and
- establishing the global well-posedness in a low regularity setting, by applying the wave packet testing method of Ifrim-Tataru (see for instance [84, 86]).

We anticipate that our streamlined analysis will open the way to substantial simplifications and improvements in the analysis of related equations, including the generalized SQG family (3.1.2).

3.1.1 Main results

We recall for the purpose of comparison the energy estimate in [6] for (3.1.3),

$$\frac{d}{dt}E^{(s)}(\varphi) \lesssim_A AB_0 \cdot E^{(s)}(\varphi), \tag{3.1.6}$$

where

$$A = \|\partial_x \varphi\|_{L^\infty}, \qquad B_0 = \|\partial_x \varphi\|_{C^{1,\delta}}.$$

We remark that this energy estimate is cubic, which is consistent with the fact that the nonlinearity Q of (3.1.3) is fully nonlinear, and cubic at leading order.

However, from the perspective of the balance of derivatives, (3.1.6) manifests quadratically, in that a full extra derivative is absorbed solely by the B_0 control norm. This in turn can be understood by viewing the cubic nonlinearity Q as a quadratic nonlinearity with a low frequency coefficient, and motivates our choice of notation $Q = Q(f, g)$.

A key observation in the current article is that the SQG equation (3.1.3) satisfies a resonance structure, in the sense that, using the quadratic perspective, the nonlinearity can be viewed as

$$Q(f, g) \approx \Omega(\partial_x^{-1} F(f_x), g), \tag{3.1.7}$$

where the symbol of Ω is

$$\Omega(\xi_1, \xi_2) = \omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2), \qquad \omega(\xi) = 2i\xi \log|\xi|.$$

For more details, see the discussion in Section 3.6.

As a result, we have access to normal form methods which open the way to a refined energy estimate with a better balance of derivatives,

$$\frac{d}{dt} E^{(s)}(\varphi) \lesssim_A B^2 \cdot E^{(s)}(\varphi),$$

where here we have balanced the control norm

$$B := \|\partial_x \varphi\|_{C^{\frac{1}{2}+}}.$$

The gain obtained from the rebalanced energy estimate, along with its counterpart for the linearized equation, leads to our main local well-posedness result:

Theorem 3.1.1. Equation (3.1.3) is locally well-posed for initial data in $\dot{H}^{s_0} \cap \dot{H}^s$ with $s > 2$ and $s_0 < \frac{3}{2}$. Precisely, for every $R > 0$, there exists $T = T(R) > 0$ such that for any $\varphi_0 \in \dot{H}^{s_0} \cap \dot{H}^s$ with $\|\varphi_0\|_{\dot{H}^{s_0} \cap \dot{H}^s} < R$, the Cauchy problem (3.1.3) has a unique solution $\varphi \in C([0, T], \dot{H}^{s_0} \cap \dot{H}^s)$. Moreover, the solution map $\varphi_0 \mapsto \varphi$ from $\dot{H}^{s_0} \cap \dot{H}^s$ to $C([0, T], \dot{H}^{s_0} \cap \dot{H}^s)$ is continuous.

Remark 3.1.1. In order to keep the proofs simpler, we assume that the parameter A is small. This assumption can be removed with a more careful definition of the paraproduct, at the expense of having to deal with more technical details.

Remark 3.1.2. We also note that up to the endpoint, this result is sharp as long as energy estimates are concerned. In order to see this, we consider the nonlinearity $\partial_x A_\varphi v$, which appears in the linearized equation, with linearized variables v. We assume that we want to bound the map $v \to \partial_x A_\varphi v$ in L^2, and that for the time being, we can redistribute the arising derivatives in $\partial_x A_\varphi v$ as we wish. We have

$$\partial_x A_\varphi v = \partial_x \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi)|\delta|^y v \, dy,$$

and we can see that, morally speaking, we have 3 derivatives, that we want to distribute onto the two φ factors (we recall that F is quadratic). In order to achieve this by requiring as little regularity from φ as possible, it is clear that the 3 derivatives have to be distributed equally between the two factors. This means that we would need the norm $\|"|D_x|^{1+\frac{\alpha}{2}}"\varphi\|_{L^\infty_x}$ to be finite, which imposes the condition $s \geq \dfrac{3}{2}$.

Normal forms were first introduced by Shatah in [123] to study the long-time behavior of solutions to dispersive equations. However, in the quasilinear context, the normal form transformation is not readily applicable, because the resulting correction will not be bounded. Several approaches have been introduced to address this, and in the present article we primarily rely on two. First is the idea of carrying out the normal form analysis in a paradifferential fashion, which was first used by Alazard-Delort [12] in a paradiagonalization argument used to obtain Sobolev estimates for the solutions of the water waves equations in the Zakharov formulation. This approach was also used by Ifrim-Tataru [85] to obtain a new proof of the L^2 global well-posedness for Benjamin-Ono equation, proved previously by Ionescu-Kenig [89].

Second is the use of modified energies, in lieu of the direct normal form transform at the level of the equation. This approach was first introduced by Hunter-Ifrim-Tataru-Wong [71] to study long time solutions of the Burgers-Hilbert equation.

The combination of these approaches to address the low regularity theory for quasilinear models was first introduced by the first author with Ifrim-Tataru in [7] for the gravity water waves system, through the proof of *b*alanced energy estimates. Balanced energy estimates were subsequently further combined with Strichartz estimates in the context of the low regularity theory for the time-like minimal surface problem in the Minkowski space [8].

We remark that our local well-posedness threshold of $s > 2$ coincides with the result of Gancedo-Nguyen-Patel [55] for patches. However, the use of the null structure in our current work yields stronger energy estimates in the sense that our control norms A and B consist of only pointwise norms rather than L^2-based norms. This allows for further applications, including the analysis of long time behavior, which we discuss next.

We will consider global well-posedness for small and localized data. To describe localized solutions, we define the operator

$$L = x + 2t + 2t \log |D_x|,$$

which commutes with the linear flow $\partial_t - 2\log|D_x|\partial_x$, and at time $t = 0$ is simply multiplication by x. Then we define the time-dependent weighted energy space

$$\|\varphi\|_X := \|\varphi\|_{\dot{H}^{s_0} \cap \dot{H}^s} + \|L\partial_x \varphi\|_{L^2},$$

where $s > 3$ and $s_0 < 1$. To track the dispersive decay of solutions, we define the pointwise control norm

$$\|\varphi\|_Y := \||D_x|^{1-\delta}\langle D_x\rangle^{\frac{1}{2}+2\delta}\varphi\|_{L^\infty_x}.$$

Theorem 3.1.2. Consider data φ_0 with

$$\|\varphi_0\|_X \lesssim \epsilon \ll 1.$$

Then the solution φ to (3.1.3) with initial data φ_0 exists globally in time, with energy bounds

$$\|\varphi(t)\|_X \lesssim \epsilon t^{C\epsilon^2}$$

and pointwise bounds

$$\|\varphi(t)\|_Y \lesssim \epsilon\langle t\rangle^{-\frac{1}{2}}.$$

Further, the solution φ exhibits a modified scattering behavior, with an asymptotic profile $W \in H^{1-C_1\epsilon^2}(\mathbb{R})$, in a sense that will be made precise in Section 3.31.

Regarding the question of large data global well-posedness, the absence of splash singularities in the case of patches was first proved by Gancedo-Strain [57]. Recently, Kiselev-Luo [105] sharpened the criterion ruling out splash singularities. More precisely, they improved the double exponential bound of Gancedo-Strain [57] to just exponential in time.

The question of extending our results to the non-graph case is interesting and open. Our current approach does not provide an answer, as a different parametrization would be needed.

3.1.2 Outline of the chapter

This chapter is organized as follows. In Section 3.2, we establish notation and preliminaries used through the rest of the chapter, including estimates involving the paradifferential calculus and difference quotients.

In Section 3.6, we introduce the null structure of equation (3.1.3) and its linearization,

$$\partial_t v - 2\log|D_x|\partial_x v = \partial_x Q(\varphi, v). \tag{3.1.8}$$

We also introduce the paradifferential flow associated to (3.1.8), which will play a central role in the subsequent analysis.

In Section 3.11, we reduce the energy estimates and well-posedness of the linearized equation (3.1.8) to that of the inhomogeneous paradifferential flow. The primary difficulty is ensuring that the paradifferential errors satisfy balanced cubic estimates. In order to achieve this, we carry out a paradifferential normal form analysis to remove the unbalanced components of the errors.

In Section 3.14, we establish energy estimates for the paradifferential flow. Here, we construct a quasilinear modified energy, which reproduces the one introduced by Hunter-Shu-Zhang in [77]. However, in the current article, we carefully observe cancellations which ensure that our estimates are balanced.

In Section 3.17, we establish higher order energy estimates. Extra care must be taken because the commutators are quadratic rather than cubic, and thus not perturbative. In order to eliminate them, we use an exponential Jacobian conjugation combined with a normal form correction.

In Section 3.20, we prove Theorem 3.1.1, the local well-posedness result for (3.1.3). We use the method of frequency envelopes to construct rough solutions as the unique limit of smooth solutions. This method was introduced by Tao in [135] to better track the evolution of energy distribution between dyadic frequencies. A systematic presentation of the use of frequency envelopes in the study of local well-posedness theory for quasilinear problems can be found in the expository paper [87].

In Section 3.21 we use the wave packet testing method of Ifrim-Tataru to prove the global-wellposedness part of Theorem 3.1.2, along with the dispersive bounds for the resulting solution. This method was systematically presented in [86]. Finally, in Section 3.31 we discuss the modified scattering behavior of the global solutions constructed in Section 3.21.

3.2 Notations and classical estimates

In this section we discuss some notation and classical estimates that we use throughout the article. These include estimates involving the paradifferential calculus, and difference quotients.

3.2.1 Paradifferential operators and paraproducts

Let χ be an even smooth function such that $\chi = 1$ on $[-\frac{1}{20}, \frac{1}{20}]$ and $\chi = 0$ outside $[-\frac{1}{10}, \frac{1}{10}]$, and define

$$\tilde{\chi}(\theta_1, \theta_2) = \chi\left(\frac{|\theta_1|^2}{M^2 + |\theta_2|^2}\right).$$

Given a symbol $a(x, \eta)$, we use the above cutoff symbol $\tilde{\chi}$ to define an M dependent paradifferential quantization of a by (see also [11])

$$\widehat{T_a u}(\xi) = (2\pi)^{-1} \int \hat{P}_{>M}(\xi)\tilde{\chi}\left(\xi - \eta, \xi + \eta\right) \hat{a}(\xi - \eta, \eta)\hat{P}_{>M}(\eta)\hat{u}(\eta)\, d\eta,$$

where the Fourier transform of the symbol $a = a(x, \eta)$ is taken with respect to the first argument.

This quantization was employed in [6], where the parameter M was introduced to ensure the coercivity of the modified quasilinear energy without relying on a small data assumption.

We recall in particular that in the case of a paraproduct, where $a = a(x)$ is real-valued, T_a is self-adjoint.

The following commutator-type estimates are exact reproductions of statements from Lemmas 2.4 and 2.6 in Section 2 of [7], respectively:

Lemma 3.3 (Para-commutators)**.** Assume that $\gamma_1, \gamma_2 < 1$. Then we have

$$\|T_f T_g - T_g T_f\|_{\dot{H}^s \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{BMO} \||D|^{\gamma_2} g\|_{BMO}, \tag{3.3.1}$$

$$\|T_f T_g - T_g T_f\|_{\dot{B}^s_{\infty,\infty} \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{L^2} \||D|^{\gamma_2} g\|_{BMO}. \tag{3.3.2}$$

A bound similar to (3.3.1) holds in the Besov scale of spaces, namely from $\dot{B}^s_{p,q}$ to $\dot{B}^{s+\gamma_1+\gamma_2}_{p,q}$ for real s and $1 \le p, q \le \infty$.

The next paraproduct estimate, see Lemma 2.5 in [7], directly relates multiplication and paramultiplication:

Lemma 3.4 (Para-products)**.** Assume that $\gamma_1, \gamma_2 < 1$, $\gamma_1 + \gamma_2 \ge 0$. Then

$$\|T_f T_g - T_{fg}\|_{\dot{H}^s \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{BMO} \||D|^{\gamma_2} g\|_{BMO}. \tag{3.4.1}$$

A similar bound holds in the Besov scale of spaces, namely from $\dot{B}^s_{p,q}$ to $\dot{B}^{s+\gamma_1+\gamma_2}_{p,q}$ for real s and $1 \le p, q \le \infty$.

Next, we recall the following Moser-type estimate. See for instance [6].

Theorem 3.4.1 (Moser)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function with $F(0) = 0$, and

$$R(v) = F(v) - T_{F'(v)} v.$$

Then

$$\|R(v)\|_{W^{\frac{1}{2},\infty}} \lesssim_{\|v\|_{L^\infty}} \||D|^{\frac{1}{2}} v\|_{L^\infty}. \tag{3.4.2}$$

3.4.1 Difference quotients

We recall that we denote difference quotients by

$$\delta^y h(x) = \frac{h(x+y) - h(x)}{y}, \qquad |\delta|^y h(x) = \frac{h(x+y) - h(x)}{|y|},$$

as well as the smooth function

$$F(s) = 1 - \frac{1}{\sqrt{1+s^2}},$$

which in particular vanishes to second order at $s = 0$, satisfying $F(0) = F'(0) = 0$. Using this notation, we may express

$$Q(\varphi, v)(t, x) = \int F(\delta^y \varphi(t, x)) \cdot |\delta|^y v(t, x)\, dy.$$

In addition, to facilitate the normal form analysis in later sections, we denote

$$\psi := \partial_x^{-1} F(\varphi_x).$$

We have the following estimate which allows the balancing of up to one derivative over multilinear averages of difference quotients:

Lemma 3.5. Let $i = \overline{1, n}$ and $p_i, r \in [1, \infty]$ and $\alpha_i, \beta_i \in [0, 1]$ satisfying

$$\sum_i \frac{1}{p_i} = \frac{1}{r}, \qquad n - 1 < \sum_i \alpha_i \leq n, \qquad 0 \leq \sum_i \beta_i < n - 1.$$

Then

$$\left\| \int \prod \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \prod \| |D|^{\alpha_i} f_i \|_{L^{p_i}} + \prod \| |D|^{\beta_i} f_i \|_{L^{p_i}}.$$

Proof. We write

$$\int \prod \delta^y f_i \, dy = \int_{|y| \leq 1} + \int_{|y| > 1}.$$

For the former integral, we have by Hölder

$$\left\| \int_{|y| \leq 1} \prod \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \int_{|y| \leq 1} \frac{1}{|y|^{n - \sum \alpha_i}} \prod \| |D|^{\alpha_i} f_i \|_{L^{p_i}} \, dy \lesssim \prod \| |D|^{\alpha_i} f_i \|_{L^{p_i}}.$$

The latter integral is treated similarly. □

3.6 The null structure and paradifferential equation

In this section and the next, we will reduce the energy estimates and well-posedness for the linearized equation (3.1.8),

$$\partial_t v - 2 \log |D_x| \partial_x v = \partial_x Q(\varphi, v),$$

to that of a paradifferential equation.

One can achieve this by viewing (3.1.8) as a paradifferential equation with perturbative source, where the main task is to paralinearize the cubic term $\partial_x Q(\varphi, v)$. Such an analysis was performed by Hunter-Shu-Zhang in [77] and refined by the authors in [6].

In the current article however, we are interested in further refining the paralinearization of (3.1.8) by insisting that the perturbative errors satisfy *b*alanced estimates. Precisely, we establish all of our estimates using only the control norms

$$A := \|\partial_x \varphi\|_{L^\infty}, \qquad B := \|\partial_x \varphi\|_{C^{\frac{1}{2}+}},$$

where A corresponds to the scaling-critical threshold, while B lies half a derivative above scaling. For comparison, the local well-posedness analysis in [6] uses control norms with $\varphi \in C^{2+}$, a full derivative above scaling.

A direct estimate of the paralinearization errors will no longer suffice to establish estimates controlled by A and B. Instead, we will rely on an appropriate paradifferential normal form transformation to remove source components that do not directly satisfy the desired balanced cubic estimates. In this section, we first consider various formulations of the paradifferential equation which will be useful in the following sections.

3.6.1 Null structure

Although $F(\delta^y \varphi)$ is principally quadratic in φ (and thus $Q(\varphi, v)$ is cubic), estimates on derivatives of $F(\delta^y \varphi)$ do not fully recognize its quadratic structure. This is because they are limited by the cases of low-high interaction where derivatives fall on the high frequency variable. As a result, from the perspective of establishing balanced estimates, $F(\delta^y \varphi)$ behaves essentially like a linear coefficient.

However, we observe that Q exhibits a null structure in the following sense. By writing

$$\Omega(f, g) = \int \delta^y f \cdot |\delta|^y g \, dy$$

and using the heuristic approximation

$$F(\delta^y \varphi) \approx T_{F'(\varphi_x)} \delta^y \varphi,$$

we may express Q as a quadratic form with low frequency coefficient,

$$Q(\varphi, v) \approx T_{F'(\varphi_x)} \Omega(\varphi, v). \tag{3.6.1}$$

We then observe that the bilinear form $\Omega(\varphi, v)$ exhibits a null structure, since its symbol

$$\Omega(\xi_1, \xi_2) = \int \operatorname{sgn} y \cdot \frac{(e^{i\xi_1 y} - 1)(e^{i\xi_2 y} - 1)}{y^2} \, dy$$

satisfies the resonance identity

$$\Omega(\xi_1, \xi_2) = \omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2), \qquad \omega(\xi) = 2i\xi \log |\xi|. \tag{3.6.2}$$

This null structure underlies the normal form analysis, which we perform in the next section.

We make the above discussion precise in the following lemma. Recall that we denote

$$\psi := \partial_x^{-1} F(\varphi_x).$$

Lemma 3.7. We have

$$Q(\varphi, v) = \Omega(\psi, v) + R(x, D)v$$

where

$$\|(\partial_x R)(x, D)v\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}.$$

Proof. We write

$$Q(\varphi, v) - \Omega(\psi, v) = \int \frac{F(\delta^y \varphi) - \delta^y \partial_x^{-1} F(\varphi_x)}{|y|} \cdot (v(x+y) - v(x))\, dy =: R(x, D)v \tag{3.7.1}$$

where

$$r(x, \xi) = \int \frac{F(\delta^y \varphi) - \delta^y \partial_x^{-1} F(\varphi_x)}{|y|} (e^{i\xi y} - 1)\, dy.$$

Then we have

$$\begin{aligned} (\partial_x R)(x, D)v &= \int \frac{F'(\delta^y \varphi)\delta^y \varphi_x - \delta^y F(\varphi_x)}{|y|} \cdot (v(x+y) - v(x))\, dy \\ &=: \int K(x, y) \cdot (v(x+y) - v(x))\, dy. \end{aligned}$$

We first estimate K, which we may write as

$$\begin{aligned} |y| K(x, y) &= \frac{1}{y}(F'(b)(a-b) - (F(a) - F(b)) + \frac{1}{y}(F'(\delta^y \varphi) - F'(\varphi_x))(a-b) \\ &=: |y| K_1(x, y) + |y| K_2(x, y), \end{aligned}$$

where $a = \varphi_x(x+y)$, $b = \varphi_x(x)$. From K_1 we obtain a Taylor expansion,

$$\|K_1(\cdot, y)\|_{L^\infty_x} \lesssim_A \left\| \frac{a-b}{y} \right\|_{L^\infty_x}^2 = \|\delta^y \varphi_x\|_{L^\infty_x}^2.$$

For K_2, we have

$$\|K_2(\cdot, y)\|_{L^\infty_x} \lesssim_A \left\| \frac{\varphi(x+y) - \varphi(x) - y\varphi_x(x)}{y^2} \right\|_{L^\infty_x} \left\| \frac{a-b}{y} \right\|_{L^\infty_x} = \|\delta^{y,(2)} \varphi\|_{L^\infty_x} \|\delta^y \varphi_x\|_{L^\infty_x},$$

where $\delta^{y,(2)}$ denotes the second-order difference quotient.

By Minkowski's inequality,

$$\begin{aligned} \|(\partial_x R)(x, D)v\|_{L^2} &\lesssim \int \|K(\cdot, y)\|_{L^\infty_x} \|v(\cdot + y) - v(\cdot)\|_{L^2_x}\, dy \\ &\lesssim_A \int \|\delta^y \varphi_x\|_{L^\infty_x} (\|\delta^y \varphi_x\|_{L^\infty_x} + \|\delta^{y,(2)} \varphi\|_{L^\infty_x}) \|v\|_{L^2_x}\, dy. \end{aligned}$$

Applying the argument of Lemma 3.5 with $\alpha_1 = \alpha_2 = \frac{1}{2}+$ and $\beta_1 = \beta_2 = \frac{1}{2}-$, we conclude

$$\|(\partial_x R)(x, D)v\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}.$$

$\square$

3.7.1 The paradifferential flow

Associated to the linearized equation (3.1.8), we have the corresponding inhomogeneous paradifferential flow,

$$\partial_t v - 2\log|D_x|\partial_x v - \partial_x Q_{lh}(\varphi, v) = f, \tag{3.7.2}$$

where we have expressed the frequency decomposition of the (essentially) quadratic form as

$$\begin{aligned} Q(\varphi, v) &= \int T_{F(\delta^y\varphi)}|\delta|^y v\, dy + \int T_{|\delta|^y v} F(\delta^y\varphi)\, dy + \int \Pi(|\delta|^y v, F(\delta^y\varphi))\, dy \\ &=: Q_{lh}(\varphi, v) + Q_{hl}(\varphi, v) + Q_{hh}(\varphi, v). \end{aligned} \tag{3.7.3}$$

We frequency decompose $\Omega = \Omega_{lh} + \Omega_{hl} + \Omega_{hh}$ in the analogous way.

In Section 3.11, we show that the linearized equation (3.1.8) reduces to the paradifferential flow (3.7.2), with f playing a perturbative role, in the sense that it satisfies balanced, cubic estimates. However, since Q and hence its paradifferential errors $Q_{hl}(\varphi, v)$ and $Q_{hh}(\varphi, v)$ are essentially quadratic, this will become apparent only after an appropriate paradifferential normal form change of variables.

Here, we establish a preliminary quadratic estimate for the reduction, which will be useful in the course of constructing and evaluating the normal form transformation later in Section 3.11.

We first extract the principal components of Ω_{lh}, which include a transport term of order 0 and a dispersive term of logarithmic order:

Lemma 3.8. We can express

$$\Omega_{lh}(\psi, v) = 2(T_{\log|D_x|\partial_x\psi} v - T_{\partial_x\psi}\log|D_x|v + \partial_x[T_\psi, \log|D_x|]v). \tag{3.8.1}$$

Further, we have

$$Q_{lh}(\varphi, v) = 2(T_{(\log|D_x|-1)\partial_x\psi + R} v - T_{\partial_x\psi}\log|D_x|v) + \Gamma(\partial_x^2\psi, \partial_x^{-1}v) \tag{3.8.2}$$

where

$$\||D_x|^{\frac{1}{2}}\Gamma\|_{L^2} \lesssim_A B\|v\|_{L^2}, \qquad \|\partial_x\Gamma\|_{L^2} \lesssim_A B\||D|^{\frac{1}{2}}v\|_{L^2}, \tag{3.8.3}$$

as well the pointwise estimates

$$\||D_x|^{\frac{1}{2}}\Gamma\|_{L^\infty} \lesssim_A B\|v\|_{L^\infty}, \qquad \|\partial_x\Gamma\|_{L^\infty} \lesssim_A B\||D|^{\frac{1}{2}}v\|_{L^\infty}. \tag{3.8.4}$$

Proof. We use the resonance identity (3.6.2) to expand

$$\begin{aligned} \Omega_{lh}(\psi, v) &= 2T_{\log|D_x|\partial_x\psi} v + 2[T_\psi, \log|D_x|\partial_x]v \\ &= 2T_{\log|D_x|\partial_x\psi} v - 2T_{\partial_x\psi}\log|D_x|v + 2\partial_x[T_\psi, \log|D_x|]v, \end{aligned} \tag{3.8.5}$$

obtaining (3.8.1). Then the remaining commutator may be expressed as

$$-2T_{\partial_x\psi}v + \Gamma(\partial_x^2\psi, \partial_x^{-1}v)$$

and Γ denotes the subprincipal remainder, which has a favorable balance of derivatives on the low frequency and thus may be estimated as (3.8.3) and (3.8.4). Combined with the low-high component of Lemma 3.7, we obtain (3.8.2). $\square$

Proposition 3.9. Consider a solution v to (3.1.8). Then v satisfies

$$(\partial_t - 2T_{1-\partial_x\psi}\log|D_x|\partial_x - 2T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x)v = f \tag{3.9.1}$$

where

$$\||D|^{-\frac{1}{2}}f\|_{L^2} \lesssim_A B\|v\|_{L^2}. \tag{3.9.2}$$

Proof. We express (3.1.8) in terms of the paradifferential equation (3.7.2) with source,

$$\partial_t v - 2\log|D_x|\partial_x v - \partial_x Q_{lh}(\varphi, v) = \partial_x Q_{hl}(\varphi, v) + \partial_x Q_{hh}(\varphi, v).$$

We estimate the source terms. We see directly from definition that $\partial_x Q_{hl}(\varphi, v)$ has a favorable balance of derivatives which satisfies (3.9.2) and may be absorbed into f. The balanced Q_{hh} term similarly satisfies (3.9.2), so we have thus reduced (3.1.8) to (3.7.2).

It then suffices to apply (3.8.2) of Lemma 3.8 to the remaining paradifferential Q_{lh} term on the left hand side of (3.7.2) to obtain (3.9.1). Here, the Γ contribution may be absorbed into f directly. Further, we have commuted the ∂_x outside Q_{lh} through the low frequency paracoefficients, since the cases where this derivative falls on the low frequency coefficients,

$$2(T_{(\log|D_x|-1)\partial_x^2\psi+\partial_x R}v - T_{\partial_x^2\psi}\log|D_x|v),$$

have a favorable balance of derivatives, satisfying (3.9.2).

$\square$

3.9.1 Nonlinear equations

We will also use the paradifferential equation (3.9.1) in the context of the nonlinear solutions φ. To conclude this section, we establish preliminary quadratic bounds on the inhomogenity of the paradifferential flow, in analogy with the preceding Proposition 3.9 for the linearized counterpart.

Proposition 3.10. Consider a solution φ to (3.1.3). Then φ satisfies

$$(\partial_t - 2T_{1-\partial_x\psi}\log|D_x|\partial_x - 2T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x)\varphi = f \tag{3.10.1}$$

where

$$\||D_x|^{\frac{1}{2}}f\|_{L^\infty} \lesssim_A B, \qquad \|\partial_x f\|_{L^\infty} \lesssim_A B^2. \tag{3.10.2}$$

The same holds for ψ in the place of φ.

Proof. We first consider the case of φ. We paradifferentially decompose $Q(\varphi, \partial_x\varphi)$ in (3.1.3) to write it in terms of the paradifferential equation (3.7.2) with source,

$$\partial_t\varphi - 2\log|D_x|\partial_x\varphi - Q_{lh}(\varphi, \partial_x\varphi) = Q_{hl}(\varphi, \partial_x\varphi) + Q_{hh}(\varphi, \partial_x\varphi).$$

As with the linearized equation, we estimate the source terms. We see directly from definition that $Q_{hl}(\varphi, \partial_x\varphi)$ has a favorable balance of derivatives which satisfies (3.10.2) and may be absorbed into f. The balanced Q_{hh} term similarly satisfies (3.10.2), so we have thus replaced $Q(\varphi, \partial_x\varphi)$ in (3.1.3) with $Q_{lh}(\varphi, \partial_x\varphi)$. In turn, it then suffices to apply Lemma 3.8 to obtain (3.10.1).

We next reduce the equation for $T_{F'(\varphi_x)}\varphi$ to that of φ. It suffices to apply the paracoefficient $T_{F'(\varphi_x)}$ to (3.10.1), and estimate the commutators. This is straightforward for the spatial paradifferential terms, applying Lemma 3.3 and observing a favorable balance of derivatives.

For the time derivative, we substitute (3.10.1) for the time derivative of φ:

$$T_{F''(\varphi_x)\partial_x\partial_t\varphi}\varphi = T_{F''(\varphi_x)(2\partial_x T_{1-\partial_x\psi}\log|D_x|\partial_x\varphi + 2\partial_x T_{(\log|D_x|-1)\partial_x\psi + R}\partial_x\varphi + \partial_x f)}\varphi.$$

The estimate (3.10.2) on $\partial_x f$ in the paracoefficient implies that its contribution in this context also satisfies (3.10.2). For the remaining terms, the favorable balance of derivatives, with two or more derivatives on the low frequency paracoefficient, again implies that we may absorb their contribution into f.

To conclude the proof for ψ, it suffices to apply the Moser estimate of Theorem 3.4.1, other than for the time derivative, for which we need to estimate

$$\partial_x^{-1}(F'(\varphi_x)\partial_x\partial_t\varphi) - \partial_t T_{F'(\varphi_x)}\varphi.$$

We decompose this into

$$[\partial_x^{-1}, T_{F'(\varphi_x)}]\partial_x\partial_t\varphi$$

which we estimate directly, using the favorable balance of derivatives, and

$$\partial_x^{-1}T_{\partial_x\partial_t\varphi}F'(\varphi_x) + \partial_x^{-1}\Pi(\partial_x\partial_t\varphi, F'(\varphi_x))$$

which is similar to the time derivative commutation in the previous reduction.

□

3.11 Reduction to the paradifferential equation

Our objective in this section is to reduce the energy estimates and well-posedness of the linearized equation (3.1.8) to that of the inhomogeneous paradifferential equation (3.7.2),

$$\partial_t v - 2\log|D_x|\partial_x v - \partial_x Q_{lh}(\varphi, v) = f.$$

To ensure that the energy estimates are balanced, we require that the inhomogeneity f satisfies balanced cubic estimates.

However, the paradifferential errors $Q_{hl}(\varphi, v)$ and $Q_{hh}(\varphi, v)$ are essentially quadratic rather than cubic, and in particular do not satisfy balanced cubic estimates. On the other hand, we saw in (3.6.1) that up to leading order and a low frequency coefficient, Q is the quadratic form associated to the resonance function for the dispersion relation of (3.1.3). This motivates the normal form change of variables

$$\tilde{v} = v - \partial_x(\psi v), \qquad \psi = \partial_x^{-1} F(\varphi_x). \tag{3.11.1}$$

However, (3.11.1) suffers from two shortcomings:

1. We cannot make use of (3.11.1) directly, as it is an unbounded transformation, and
2. quartic (essentially cubic) residuals in the equation for $\tilde{v}$ given by (3.11.1) are still unbalanced.

To address the first shortcoming, we instead consider a bounded paradifferential component of (3.11.1),

$$\tilde{v} = v - \partial_x(T_v \psi) - \partial_x \Pi(v, \psi) \tag{3.11.2}$$

which is compatible with our objective of reducing to the paradifferential equation (3.7.2). To address the second shortcoming, we refine (3.11.2) with a low frequency Jacobian coefficient which addresses the quartic and higher order residuals:

$$\tilde{v} = v - \partial_x T_{T_J v} \psi - \partial_x \Pi(T_J v, \psi), \qquad J = (1 - \partial_x \psi)^{-1}. \tag{3.11.3}$$

Proposition 3.12. Consider a solution v to (3.1.8). Then we have

$$\partial_t \tilde{v} - 2 \log |D_x| \partial_x \tilde{v} - \partial_x Q_{lh}(\varphi, \tilde{v}) = \tilde{f}, \tag{3.12.1}$$

where $\tilde{f}$ satisfies balanced cubic estimates,

$$\|\tilde{f}\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}. \tag{3.12.2}$$

Proof. We express v satisfying (3.1.8) in terms of the paradifferential equation (3.7.2) with source,

$$\partial_t v - 2 \log |D_x| \partial_x v - \partial_x Q_{lh}(\varphi, v) = \partial_x Q_{hl}(\varphi, v) + \partial_x Q_{hh}(\varphi, v).$$

Unlike in Proposition 3.9, we do not estimate the source terms directly. Instead, we will establish the following cancellation with the contribution from the normal form correction,

$$\partial_t \partial_x T_{T_J v} \psi - 2 \log |D_x| \partial_x^2 T_{T_J v} \psi - \partial_x Q_{lh}(\psi, \partial_x T_{T_J v} \psi) = \partial_x Q_{hl}(\varphi, v) + \tilde{f}, \tag{3.12.3}$$

with the analogous relationship for the balanced Π component of the correction, with Q_{hh}.

To show (3.12.3), we first observe that using Lemma 3.8, we may replace $\partial_x Q_{lh}$ on the left hand side of (3.12.3) by its principal components. The Γ error is estimated using the second estimate of (3.8.3),

$$\|\partial_x \Gamma\|_{L^2} \lesssim_A B \||D|^{\frac{1}{2}} \partial_x T_{T_J v} \psi\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}$$

and may be absorbed into $\tilde{f}$. It thus suffices to show

$$(\partial_t - 2\partial_x (T_{1-\partial_x \psi} \log |D_x| - T_{(\log |D_x| - 1)\partial_x \psi + R}))\partial_x T_{T_J v} \psi = \partial_x Q_{hl}(\varphi, v) + \tilde{f}. \tag{3.12.4}$$

Next, we compute the time derivative in (3.12.4). The case where ∂_t falls on the low frequency J may be absorbed into $\tilde{f}$ due to a favorable balance of derivatives. More precisely, we use (3.10.1) to write

$$\partial_t J = J^2 \partial_x \partial_t \psi = J^2 \partial_x (2T_{1-\partial_x \psi} \log |D_x| \partial_x \psi + 2T_{(\log |D_x| - 1)\partial_x \psi + R} \partial_x \psi + f)$$

so that we can we can estimate for instance the contribution of the first term,

$$\|\partial_x T_{T_{2J^2 \partial_x^2 \log |D_x| \psi} v} \psi\|_{L^2} \lesssim_A B^2 \|v\|_{L^2},$$

with similar estimates for the other contributions, using the first estimate of (3.10.2) for the contribution of f.

In the remaining cases, ∂_t falls on the middle frequency v or the high frequency ψ, so we use (3.9.1) and (3.10.1) respectively to write

$$\begin{aligned} &\partial_x T_{T_J \partial_t v} \psi = \partial_x T_{T_J (2T_{1-\partial_x \psi} \log |D_x| \partial_x v + 2T_{(\log |D_x| - 1)\partial_x \psi + R} \partial_x v + f)} \psi, \\ &\partial_x T_{T_J v} \partial_t \psi = \partial_x T_{T_J v} (2T_{1-\partial_x \psi} \log |D_x| \partial_x \psi + 2T_{(\log |D_x| - 1)\partial_x \psi + R} \partial_x \psi + f_\psi). \end{aligned} \tag{3.12.5}$$

We consider the first, second, and third contributions from the two equations of (3.12.5) in pairs:

i) The first terms in (3.12.5) combine with the second term on the left in (3.12.4),

$$2\partial_x (T_{T_J T_{1-\partial_x \psi} \log |D_x| \partial_x v} \psi + T_{T_J v} T_{1-\partial_x \psi} \log |D_x| \partial_x \psi - T_{1-\partial_x \psi} \log |D_x| \partial_x T_{T_J v} \psi), \tag{3.12.6}$$

to form $\partial_x Q_{hl}(\varphi, v)$, modulo balanced errors which may be absorbed into $\tilde{f}$. To see this, we will use in each of the three terms that $(1 - \partial_x \psi)J = 1$. As we do so, we need to take care that any paraproduct errors and commutators yield balanced errors.

First, we observe that in the third term, we can apply the commutator estimate

$$\|\partial_x [T_{1-\partial_x \psi}, \log |D_x| \partial_x] T_{T_J v} \psi\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}.$$

Then using Lemma 3.4 to compose paraproducts in each of the three terms of (3.12.6), we have

$$2\partial_x (T_{\log |D_x| \partial_x v} \psi + T_{(1-\partial_x \psi) T_J v} \log |D_x| \partial_x \psi - \log |D_x| \partial_x T_{(1-\partial_x \psi) T_J v} \psi).$$

For the latter two terms, we will also use Lemma 3.4 to compose paraproducts, before applying $(1-\partial_x\psi)J = 1$. To do so, we first need to exchange multiplication by $(1-\partial_x\psi)$ with the paraproduct $T_{1-\partial_x\psi}$. However, the error from this exchange is not directly perturbative. Instead, we perform the exchange for the two terms simultaneously, to observe a cancellation in the form of the commutator

$$2\partial_x(T_{T_{T_Jv}(1-\partial_x\psi)} \log|D_x|\partial_x\psi - \log|D_x|\partial_x T_{T_{T_Jv}(1-\partial_x\psi)}\psi),$$

which has a favorable balance of derivatives and may be absorbed into $\tilde{f}$. The same holds for the analogous cases with $\Pi(1-\partial_x\psi, T_Jv)$. We have thus reduced (3.12.6) to

$$2\partial_x(T_{\log|D_x|\partial_x v}\psi + T_v \log|D_x|\partial_x\psi - \log|D_x|\partial_x T_v\psi) = \partial_x\Omega_{hl}(\psi, v)$$

which by Lemma 3.7 coincides with $\partial_x Q_{hl}(\varphi, v)$ up to balanced errors, as desired.

ii) The second terms in (3.12.5),

$$2\partial_x(T_{T_J T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x v}\psi + T_{T_Jv}T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x\psi), \tag{3.12.7}$$

combine to cancel the third term on the left hand side of (3.12.4), up to balanced errors. To see this, we apply the commutator Lemma 3.3 to exchange the first term of (3.12.7) with

$$2\partial_x T_{T_{(\log|D_x|-1)\partial_x\psi+R}T_J\partial_x v}\psi.$$

We can freely exchange the low frequency paraproduct $T_{(\log|D_x|-1)\partial_x\psi+R}$ with a standard product, since

$$\|\partial_x T_{T_{T_J\partial_x v}(\log|D_x|-1)\partial_x\psi + T_{T_J\partial_x v}R}\psi\|_{L^2} \lesssim_A B^2\|v\|_{L^2} \tag{3.12.8}$$

and likewise for the balanced Π case. We thus have

$$2\partial_x T_{T_J\partial_x v\cdot(\log|D_x|-1)\partial_x\psi+R}\psi.$$

Then applying Lemma 3.4 for splitting paraproducts, and returning to (3.12.7), we arrive at

$$2\partial_x(T_{T_J\partial_x v}T_{(\log|D_x|-1)\partial_x\psi+R}\psi + T_{T_Jv}T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x\psi).$$

Lastly, we factor out a derivative,

$$2\partial_x^2 T_{T_Jv}T_{(\log|D_x|-1)\partial_x\psi+R}\psi$$

where we have absorbed the cases where the factored derivative falls on J or $(\log|D_x|-1)\partial_x\psi+R$ into $\tilde{f}$, similar to (3.12.8). After one more instance of the commutator Lemma 3.3, we arrive at the third term on the left hand side of (3.12.4) as desired.

iii) By Propositions 3.9 and 3.10 respectively, the contributions from f and f_ψ satisfy (3.12.2) and may be absorbed into $\tilde{f}$.

□

We also obtain a similar but easier balanced estimate for the reduction of the nonlinear equation to the paradifferential flow, in the $\dot{H}^s$ setting. Here the normal form correction consists only of a balanced Π component:

$$\tilde{\varphi} = \varphi - \Pi(\psi, T_J \partial_x \varphi). \tag{3.12.9}$$

Proposition 3.13. Consider a solution φ to (3.1.3). Then we have

$$\partial_t \tilde{\varphi} - 2 \log |D_x| \partial_x \tilde{\varphi} - \partial_x Q_{lh}(\varphi, \tilde{\varphi}) = \tilde{f}, \tag{3.13.1}$$

where $\tilde{f}$ satisfies balanced cubic estimates,

$$\|\tilde{f}\|_{\dot{H}^s} \lesssim_A B^2 \|\varphi\|_{\dot{H}^s_x}. \tag{3.13.2}$$

Proof. First observe that we have

$$\partial_t \varphi - 2 \log |D_x| \partial_x \varphi - \partial_x Q_{lh}(\varphi, \varphi) = Q_{hh}(\varphi, \partial_x \varphi).$$

Then the normal form analysis is similar to the analysis for the (balanced) paradifferential errors of the linear equation in Proposition 3.12.

To see that we can obtain balanced estimates in the $\dot{H}^s$ setting, first observe that in each of the estimates in the proof of Proposition 3.12, we can easily obtain at least one B from the estimate of the low frequency variable. Then since we are in the balanced Π setting, we can obtain a second B, with s outstanding derivatives, which can then be placed on the remaining high frequency factor.

□

3.14 Energy estimates for the paradifferential equation

In this section we establish energy estimates for the paradifferential equation (3.7.2). We define the modified energy

$$E(v) := \int v \cdot T_{1-\partial_x \psi} v \, dx.$$

Proposition 3.15. We have

$$\frac{d}{dt} E(v) \lesssim_A B^2 \|v\|_{L^2}^2 + \|f\|_{L^2} \|v\|_{L^2}. \tag{3.15.1}$$

Proof. Without loss of generality we assume $f = 0$. We consider first the linear component of the energy. Using the equation (3.7.2) for v and skew adjointness of $\log |D_x| \partial_x$, we have

$$\frac{1}{2} \frac{d}{dt} \int v \cdot v \, dx = \int (2 \log |D_x| \partial_x v + \partial_x Q_{lh}(\varphi, v)) \cdot v \, dx = \int \partial_x Q_{lh}(\varphi, v) \cdot v \, dx.$$

Using Lemma 3.8 to expand Q_{lh}, this may be written

$$\int 2\partial_x(T_{\log|D_x|\partial_x\psi+R}v + T_\psi \log|D_x|\partial_x v - \log|D_x|\partial_x T_\psi v) \cdot v\, dx.$$

Cyclically integrating by parts (the first term individually, and the latter two terms paired), this may be expressed as

$$\int (T_{\partial_x^2 \log|D_x|\psi+\partial_x R}v + 2T_{\partial_x\psi}\log|D_x|\partial_x v) \cdot v\, dx. \tag{3.15.2}$$

Here, the contribution with $\partial_x R$ is balanced by Lemma 3.7 and may be discarded.

Next we evaluate the effect of the quasilinear modification to the energy, using (3.7.2) and (3.10.1) respectively to expand time derivatives:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int v \cdot T_{\partial_x\psi}v\, dx &= \int \partial_t v \cdot T_{\partial_x\psi}v\, dx + \frac{1}{2}\int v\cdot T_{\partial_x\partial_t\psi}v\, dx \\ &= \int (2\log|D_x|\partial_x v + \partial_x Q_{lh}(\varphi, v)) \cdot T_{\partial_x\psi}v\, dx \\ &\quad + \frac{1}{2}\int v \cdot T_{\partial_x(2T_{1-\partial_x\psi}\log|D_x|\partial_x\psi + 2T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x\psi + f)}v\, dx.\end{aligned} \tag{3.15.3}$$

The contribution from f may be estimated using (3.10.2) and discarded. The cubic terms cancel with (3.15.2), so it remains to estimate the quartic terms,

$$\int \partial_x Q_{lh}(\varphi, v) \cdot T_{\partial_x\psi}v + v \cdot T_{\partial_x(T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x\psi - T_{\partial_x\psi}\log|D_x|\partial_x\psi)}v\, dx. \tag{3.15.4}$$

We expand Q_{lh} in the first term of (3.15.4), and will observe cancellations in two steps with the remaining terms. The expansion is similar to the expansion of the L^2 energy above, except with an additional $T_{\partial_x\psi}$ paraproduct:

$$\int 2\partial_x(T_{\log|D_x|\partial_x\psi+R}v + T_\psi\log|D_x|\partial_x v - \log|D_x|\partial_x T_\psi v)\cdot T_{\partial_x\psi}v\, dx. \tag{3.15.5}$$

i) From the first term in (3.15.5), we have after cyclically integrating by parts,

$$\int v\cdot T_{\partial_x^2\log|D_x|\psi+\partial_x R}T_{\partial_x\psi}v - v\cdot T_{\log|D_x|\partial_x\psi+R}T_{\partial_x^2\psi}v + v\cdot[T_{\partial_x\psi}, T_{\log|D_x|\partial_x\psi+R}]\partial_x v\, dx.$$

The commutator satisfies (3.15.1) by Lemma 3.3, up to errors also satisfying (3.15.1), as well as the non-perturbative residual

$$-\int v\cdot T_{\partial_x T_{\partial_x\psi}\partial_x\psi}v\, dx \tag{3.15.6}$$

which we will address in the next step. To see these cancellations, we use Lemma 3.4 to compose paraproducts, and observe that any contribution with two or more derivatives on the lowest frequency has a favorable balance of derivatives and satisfies (3.15.1). For instance, from the second term of (3.15.4), we have the perturbative component

$$\left| \int v \cdot T_{T_{\partial_x^2 \log|D_x|\psi} \partial_x \psi} v \, dx \right| \lesssim_A B^2 \|v\|_{L^2}.$$

ii) It remains to estimate the last two terms in (3.15.5), along with (3.15.6). In the last term of (3.15.5), the case where the ∂_x falls on the low frequency ψ vanishes by skew adjointness. From what remains, we have the commutator

$$-2 \int [T_\psi, \log|D_x|]\partial_x v \cdot \partial_x T_{\partial_x \psi} v \, dx = 2 \int T_{\partial_x \psi} \partial_x [T_\psi, \log|D_x|]\partial_x v \cdot v \, dx.$$

On the other hand, due to skew adjointness, we are free to insert

$$\int 2 T_{\partial_x \psi} \partial_x T_{\partial_x \psi} v \cdot v \, dx = 0$$

which subtracts the principal component of the commutator. In addition, we can rewrite (3.15.6), up to perturbative errors, as

$$- \int T_{\partial_x \psi} T_{\partial_x^2 \psi} v \cdot v \, dx.$$

Combined, we have

$$\int T_{\partial_x \psi} L(\partial_x^2 \psi, v) \cdot v \, dx \tag{3.15.7}$$

where

$$L(\partial_x^2 \psi, v) = (2\partial_x [T_\psi, \log|D_x|]\partial_x + 2\partial_x T_{\partial_x \psi} - T_{\partial_x^2 \psi})v.$$

Since the components

$$2\partial_x [T_\psi, \log|D_x|]\partial_x, \qquad 2\partial_x T_{\partial_x \psi} - T_{\partial_x^2 \psi}$$

of $L(\partial_x^2 \psi, \cdot)$ are both skew-adjoint, $L(\partial_x^2 \psi, \cdot)$ is skew-adjoint as well. We thus have a commutator form for (3.15.7),

$$\int T_{\partial_x \psi} L(\partial_x^2 \psi, v) \cdot v \, dx = \frac{1}{2} \int (T_{\partial_x \psi} L(\partial_x^2 \psi, v) - L(\partial_x^2 \psi, T_{\partial_x \psi} v)) \cdot v \, dx$$

for which we have the desired balanced estimate.

□

By combining this result with the normal form analysis from the previous section, we obtain the following well-posedness result:

Proposition 3.16. Assume that $A \ll 1$ and $B \in L^2_t$. There exists an energy functional $E_{lin}(v)$ such that for every solution of (3.1.8), we have the following:

a) Norm equivalence:

$$E_{lin}(v) \approx_A \|v\|^2_{L^2_x}$$

b) Energy estimates:

$$\frac{d}{dt}E_{lin}(v) \lesssim_A B^2\|v\|^2_{L^2_x}$$

Remark 3.16.1. It actually turns out that the linearized equation (3.1.8) is well-posed in L^2_x. We are not going to use this property, but we briefly discuss the key ideas behind its proof. The main point is to obtain a similar estimate for the adjoint equation, interpreted as a backward evolution in the space L^2. Namely, the adjoint equation corresponding to the linearized one has the form

$$\partial_t v - 2\log|D_x|\partial_x v - Q(\varphi, \partial_x v) = 0$$

By carrying out a paradifferential normal form transformation, akin to the one from the proof of Proposition 3.12, we reduce this to the equation

$$\partial_t v - 2\log|D_x|\partial_x v - Q_{lh}(\varphi, \partial_x v) = 0.$$

By considering the modified energy functional

$$\int v \cdot T_{\frac{1}{1-\psi_x}} v\, dx,$$

and carrying out an analysis similar to the one from the proof of Proposition 3.15, we obtain the desired energy estimate for the dual problem. Now the existence follows by a standard duality argument (for the general theory, see Theorem 23.1.2 in Hörmander [67]).

Proof. Let $E_{lin}(v) = E(\tilde{v})$, where $E(\cdot)$ is defined in Proposition 3.15, and $\tilde{v}$ is defined in Proposition 3.12.

Part a) is immediate, whereas part b) follows from Proposition 3.15. □

3.17 Higher order energy estimates

In this section we establish higher order energy estimates. Extra care must be taken because commutators with D^s are quadratic rather than cubic, and thus require a normal form correction.

Proposition 3.18. Let $s \geq 0$. Given v solving (3.12.1), there exists a normalized variable v^s such that

$$\partial_t v^s - 2\log|D_x|\partial_x v^s - \partial_x Q_{lh}(\varphi, v^s) = f + \mathscr{R}(\varphi, v),$$

with

$$\|v^s - |D_x|^s v\|_{L^2_x} \lesssim_A A\|v\|_{\dot{H}^s_x}$$

and $\mathscr{R}(\varphi)$ satisfying balanced cubic estimates,

$$\|\mathscr{R}(\varphi, v)\|_{L^2} \lesssim_A B^2\|v\|_{L^2}. \tag{3.18.1}$$

Proof. Let v satisfy (3.12.1), where without loss of generality, $f = 0$. The natural approach is to reduce the equation for $v^s := |D_x|^s v$ to (3.12.1) with a perturbative inhomogeneity. However, the commutators arising from such a reduction are quadratic, and cannot satisfy balanced cubic estimates. In particular, they cannot be seen as directly perturbative. We will address these errors via a conjugation combined with a normal form correction.

In preparation, we use Lemmas 3.7 and 3.8 to rewrite Q_{lh} in (3.12.1) and obtain

$$\partial_t v - 2\partial_x(T_{1-\partial_x\psi}\log|D_x| + T_{\log|D_x|\partial_x\psi + R} + \partial_x[T_\psi, \log|D_x|])v = 0. \tag{3.18.2}$$

Then $v^s := |D_x|^s v$ satisfies

$$\begin{aligned}
&\partial_t v^s - 2\partial_x(T_{1-\partial_x\psi}\log|D_x| + T_{\log|D_x|\partial_x\psi + R} + \partial_x[T_\psi, \log|D_x|])v^s \\
&\quad = 2\partial_x([|D_x|^s, T_{-\partial_x\psi}]\log|D_x| + [|D_x|^s, T_{\log|D_x|\partial_x\psi + R}] + \partial_x[|D_x|^s, [T_\psi, \log|D_x|]])v \\
&\quad =: L(\partial_x^2\psi, \log|D_x|v^s) - L(\log|D_x|\partial_x^2\psi, v^s) + 2\partial_x^2[|D_x|^s, [T_\psi, \log|D_x|]]v + \mathscr{R}
\end{aligned} \tag{3.18.3}$$

where we have absorbed $\partial_x R$ into $\mathscr{R}$ and L denotes an order zero paradifferential bilinear form,

$$L(\partial_x f, u) = -2\partial_x[|D_x|^s, T_f]|D_x|^{-s}u. \tag{3.18.4}$$

In particular, observe that the principal term of L is given by

$$L(g, u) \approx -2sT_g u.$$

To address the two L contributions, which are quadratic and not directly perturbative, we apply two steps:

a) We first apply a conjugation to v^s which improves the leading order of the contributions of the L terms from, up to a logarithm, 0 to -1.

b) We then apply a normal form transformation yielding cubic, balanced source terms.

a) We begin by computing the equation for the conjugated variable

$$\tilde{v}^s := T_{J^{-s}} v^s.$$

To do so, it suffices to apply $T_{J^{-s}}$ to (3.18.3) and consider the commutators. These will include a ∂_t commutator, a ∂_x commutator, and a $\log|D_x|$ commutator.

i) First, we use (3.10.1) to expand the ∂_t commutator,

$$[T_{J^{-s}}, \partial_t]v^s = -sT_{J^{1-s}\partial_x\partial_t\psi}v^s = -sT_{J^{1-s}\partial_x(2T_{1-\partial_x\psi}\log|D_x|\partial_x\psi + 2T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x\psi + f)}v^s. \tag{3.18.5}$$

Here the contribution from f may be estimated using (3.10.2) and discarded. Further, due to a favorable balance of derivatives when ∂_x falls on the lowest frequency variables, we can reduce to

$$-2sT_{J^{-s}(T_J T_{1-\partial_x\psi}\log|D_x|\partial_x^2\psi + T_J T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x^2\psi)}v^s.$$

Lastly, we apply Lemma 3.4 to merge and split paraproducts, reducing to

$$-2sT_{\log|D_x|\partial_x^2\psi + T_{J(\log|D_x|-1)\partial_x\psi+R}\partial_x^2\psi}\tilde{v}^s. \tag{3.18.6}$$

Observe that the first part of (3.18.6) cancels with the principal term of the second L on the right hand side of (3.18.3). The second part of (3.18.6) will cancel with a contribution from ii) below.

ii) Next, we consider the commutator of $T_{J^{-s}}$ with the outer ∂_x. We obtain

$$2sT_{J^{1-s}\partial_x^2\psi}(T_{1-\partial_x\psi}\log|D_x| + T_{\log|D_x|\partial_x\psi+R} + \partial_x[T_\psi, \log|D_x|])v^s.$$

Here it is convenient to apply (3.8.2) of Lemma 3.8 to write this as

$$2sT_{J^{1-s}\partial_x^2\psi}((T_{1-\partial_x\psi}\log|D_x| + T_{(\log|D_x|-1)\partial_x\psi+R})v^s + \Gamma(\partial_x^2\psi, \partial_x^{-1}v^s)).$$

Applying Lemmas 3.4 and 3.3 to split, compose, and commute paraproducts, this may be reduced modulo perturbative terms to

$$2sT_{\partial_x^2\psi}\log|D_x|\tilde{v}^s + 2sT_{J(\log|D_x|-1)\partial_x\psi\cdot\partial_x^2\psi + \partial_x^2\psi\cdot R}\tilde{v}^s.$$

The first term above cancels with the principal term of the first L. The second term cancels with the remaining part of the ∂_t commutator above in (3.18.6). To see this cancellation, we have freely exchanged multiplication by $J(\log|D_x|-1)\partial_x\psi$ with a paraproduct, as the difference has a favorable balance of derivatives and is thus perturbative.

iii) Returning to the commutator of $T_{J^{-s}}$ with the dispersive term, it remains to consider the commutator with the inner $\log|D_x|$, where we have used Lemma 3.3 to discard any paraproduct commutators. We have

$$-2\partial_x T_{1-\partial_x\psi}[T_{J^{-s}}, \log|D_x|]v^s$$

whose principal term $2sT_{\partial_x^2\psi}\tilde{v}^s$ cancels with the principal term of the double commutator on the right hand side of (3.18.3).

To conclude, we have

$$\begin{aligned}
&\partial_t\tilde{v}^s - 2\partial_x(T_{1-\partial_x\psi}\log|D_x| + T_{\log|D_x|\partial_x\psi+R} + \partial_x[T_\psi, \log|D_x|])\tilde{v}^s \\
&\quad = (L(\partial_x^2\psi, \log|D_x|\tilde{v}^s) + 2sT_{\partial_x^2\psi}\log|D_x|\tilde{v}^s) \\
&\qquad - (L(\log|D_x|\partial_x^2\psi, \tilde{v}^s) + 2sT_{\log|D_x|\partial_x^2\psi}\tilde{v}^s) \\
&\qquad + 2T_{J^{-s}}\partial_x^2[|D_x|^s, [T_\psi, \log|D_x|]]v - 2\partial_x T_{1-\partial_x\psi}[T_{J^{-s}}, \log|D_x|]v^s + f \\
&\quad =: L_0(\partial_x^3\psi, \log|D_x|\partial_x^{-1}\tilde{v}^s) - L_0(\log|D_x|\partial_x^3\psi, \partial_x^{-1}\tilde{v}^s) + L_1(\partial_x^3\psi, \partial_x^{-1}\tilde{v}^s) + f
\end{aligned} \tag{3.18.7}$$

where f satisfies (3.18.1). Here L_0 and L_1 denote order zero paradifferential bilinear forms, respectively

$$\begin{aligned}
L_0(\partial_x^2 f, \partial_x^{-1}u) &= L(\partial_x f, u) + 2sT_{\partial_x f}u, \\
L_1(\partial_x^2 f, \partial_x^{-1}T_{J^{-s}}u) &= (2T_{J^{-s}}\partial_x^2[|D_x|^s, [T_{\partial_x^{-1}f}, \log|D_x|]]|D_x|^{-s}u - 2\partial_x T_{1-\partial_x\psi}[T_{J^{-s}}, \log|D_x|]u).
\end{aligned}$$

Observe that since L_i are all order 0 paradifferential bilinear forms, we have reduced the terms of the inhomogeneity to order -1.

b) We next choose a normal form transformation to reduce the quadratic components of the inhomogeneity to balanced cubic terms. Let

$$\tilde{w}^s = \frac{1}{2}T_J L_0(\partial_x^2\psi, \partial_x^{-1}\tilde{v}^s).$$

Then we claim that $\tilde{u}^s := \tilde{v}^s - \tilde{w}^s$ is the desired normal form transform. To see this, it remains to compute

$$(\partial_t - 2\partial_x(T_{1-\partial_x\psi}\log|D_x| + T_{\log|D_x|\partial_x\psi+R} + \partial_x[T_\psi, \log|D_x|]))\tilde{w}^s \tag{3.18.8}$$

and observe cancellation with the three L_i bilinear forms on the right hand side of (3.18.7). To see this, we partition the computation into the following subgroups:

i) When the full equation of (3.18.8) falls on the high frequency $\tilde{v}^s$ input of L_0, we may use (3.18.7) to see that the contribution has a favorable balance of derivatives and may be absorbed into f.

ii) We may commute the equation freely with the low frequency J due to a favorable balance of derivatives, absorbing the contribution again into f.

It remains to consider commutators of the terms of the equation (3.18.8) across the low frequency $\partial_x^2\psi$ input of $\tilde{w}^s$.

iii) We first consider the commutators involving the operators

$$2\partial_x(T_{\log|D_x|\partial_x\psi+R} + \partial_x[T_\psi, \log|D_x|]).$$

We may freely commute the ∂_x forward, and also use (3.8.2) of Lemma 3.8 to rewrite, reducing to the operators

$$2T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x + \Gamma(\partial_x^2\psi, \partial_x^{-1}(\cdot)) \circ \partial_x.$$

The contribution from the Γ term may be absorbed into f due to a favorable balance. The remaining contribution

$$T_J L_0(T_{(\log|D_x|-1)\partial_x\psi+R}\partial_x^3\psi, \partial_x^{-1}\tilde{v}^s)$$

will cancel with a contribution of step iv) below.

iv) For the case when ∂_t falls on the low frequency input of L_0, we apply equation (3.10.1). Precisely, the two non-perturbative contributions on the left hand side of (3.10.1) cancel with the second L_0 source term in (3.18.7), and the remaining contribution of step iii) above, respectively.

v) From the dispersive term $2T_{J^{-1}}\partial_x \log|D_x|$, the case when ∂_x falls on the low frequency input of L_0 while $\log|D_x|$ has commuted to the high frequency input cancels with the first L_0 source term in (3.18.7).

vi) From the dispersive term $2T_{J^{-1}}\partial_x \log|D_x|$, it remains to consider the commutators with $\log|D_x|$, where the ∂_x remains in front. Using Lemma 3.4 with $J(1-\partial_x\psi) = 1$, and opening the definition of L_0, we have

$$2\partial_x^2[\log|D_x|, [|D_x|^s, T_\psi]]|D_x|^{-s}\tilde{v}^s - 2s\partial_x[\log|D_x|, T_{\partial_x\psi}]\tilde{v}^s.$$

We claim that these two terms cancel with the two terms of L_1 respectively. Indeed, for the first term, we commute using Lemma 3.3 to reduce to

$$2T_{J^{-s}}\partial_x^2[\log|D_x|, [|D_x|^s, T_\psi]]v$$

which cancels with the double commutator term of L_1. For the second term, we also commute using Lemma 3.3 to reduce to

$$2sT_{J^{-s}}\partial_x[T_{\partial_x\psi}, \log|D_x|]v^s =: -2sT_{J^{-s}}T_{\partial_x^2\psi}v^s + T_{J^{-s}}L_2(\partial_x^3\psi, \partial_x^{-1}v).$$

On the other hand, the second term of L_1 may be expressed as

$$-2\partial_x T_{1-\partial_x\psi}[T_{J^{-s}}, \log|D_x|]v^s = 2sT_{1-\partial_x\psi}T_{J^{1-s}\partial_x^2\psi}v^s + T_{1-\partial_x\psi}L_2(J^{1-s}\partial_x^3\psi, \partial_x^{-1}v).$$

These cancel, up to applying Lemma 3.4 with $J(1-\partial_x\psi) = 1$ and perturbative errors. $\square$

We thus obtain the following energy estimate:

Proposition 3.19. Assume that $A \ll 1$ and $B \in L_t^2$. For every $s \geq 0$, there exist energy functionals $E^{(s)}(v)$ such that we have the following:

a) Norm equivalence:

$$E^{(s)}(v) \approx_A \|v\|_{\dot{H}_x^s}^2$$

b) Energy estimates:

$$\frac{d}{dt}E^{(s)}(v) \lesssim_A B^2\|v\|_{\dot{H}_x^s}^2$$

Proof. Let $E^{(s)}(v) = E(v^s)$, where $E(\cdot)$ is defined in Proposition 3.15, and v^s is defined in Proposition 3.18.

Part a) is immediate, whereas part b) follows from Proposition 3.15. $\square$

3.20 Local well-posedness

To establish the local well-posedness result at low regularity, we follow the approach outlined in [87]. We consider $\varphi_0 \in \dot{H}_x^{s_1} \cap \dot{H}_x^{s_2}$, with $s_1 < \frac{3}{2}$, $s_2 > 2$. Let $\varphi_0^h = (\varphi_0)_{\leq h}$, where $h \in \mathbb{N}$. Since $\varphi_0^h \to \varphi_0$ in $\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2}$, we may assume that $\|\varphi_0^h\|_{\dot{H}_x^{s_1}\cap\dot{H}_x^{s_2}} < R$ for all h.

We construct a uniform $\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2}$ frequency envelope $\{c_k\}_{k\in\mathbb{Z}}$ for φ_0 having the following properties:

a) Uniform bounds:

$$\|P_k(\varphi_0^h)\|_{\dot{H}_x^{s_1}\cap\dot{H}_x^{s_2}} \lesssim c_k,$$

b) High frequency bounds:

$$\|\varphi_0^h\|_{\dot{H}_x^{s_1}\cap\dot{H}_x^{N}} \lesssim 2^{h(N-s_2)}c_h, \qquad N > s_2,$$

c) Difference bounds:

$$\|\varphi_0^{h+1} - \varphi_0^h\|_{L_x^2} \lesssim 2^{-s_2 h}c_h,$$

d) Limit as $h \to \infty$:

$$\varphi_0^h \to \varphi_0 \in \dot{H}_x^{s_1} \cap \dot{H}_x^{s_2}.$$

Let φ^h be the solutions with initial data φ_0^h, whose existence is guaranteed instance by [6]. Using the energy estimate for the solution φ of (3.1.3) from Proposition 3.19 and Proposition 3.13, we deduce that there exists $T = T(\|\varphi_0\|_{H_x^s}) > 0$ on which all of these solutions are defined, with high frequency bounds

$$\|\varphi^h\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^N)} \lesssim \|\varphi_0^h\|_{\dot{H}_x^{s_1} \cap \dot{H}_x^N} \lesssim 2^{h(N-s_2)} c_h.$$

Further, by using the energy estimates for the solution of the linearized equation from Proposition 3.16, we have

$$\|\varphi^{h+1} - \varphi^h\|_{C_t^0 L_x^2} \lesssim 2^{-s_2 h} c_h.$$

By interpolation, we infer that

$$\|\varphi^{h+1} - \varphi^h\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} \lesssim c_h.$$

As in [87], we get

$$\|P_k \varphi^h\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} \lesssim c_k$$

and that

$$\|\varphi^{h+k} - \varphi^h\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} \lesssim c_{h \leq \cdot < h+k} = \left(\sum_{n=h}^{h+k-1} c_n^2 \right)^{\frac{1}{2}}$$

for every $k \geq 1$. Thus, φ^h converges to an element φ belonging to $C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})([0,T] \times \mathbb{R})$. Moreover, we also obtain

$$\|\varphi^h - \varphi\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} \lesssim c_{\geq h} = \left(\sum_{n=h}^{\infty} c_n^2 \right)^{\frac{1}{2}}. \tag{3.20.1}$$

We now prove continuity with respect to the initial data. We consider a sequence

$$\varphi_{0j} \to \varphi_0 \in (\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})$$

and an associated sequence of H_x^s-frequency envelopes $\{c_k^j\}_{k \in \mathbb{Z}}$, each satisfying the analogous properties enumerated above for c_k, and further such that $c_k^j \to c_k$ in $l^2(\mathbb{Z})$. In particular,

$$\|\varphi_j^h - \varphi_j\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} \lesssim c_{\geq h}^j = \left(\sum_{n=h}^{\infty} (c_n^j)^2 \right)^{\frac{1}{2}}. \tag{3.20.2}$$

Using the triangle inequality with (3.20.1) and (3.20.2), we write

$$\begin{aligned} \|\varphi_j - \varphi\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} &\lesssim \|\varphi^h - \varphi\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} + \|\varphi_j^h - \varphi_j\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} + \|\varphi_j^h - \varphi^h\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})} \\ &\lesssim c_{\geq h} + c_{\geq h}^j + \|\varphi_j^h - \varphi^h\|_{C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})}. \end{aligned}$$

To address the third term, we observe that for every fixed h, $\varphi_j^h \to \varphi^h$ in $(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})$. We conclude that $\varphi_j \to \varphi$ in $C_t^0(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})([0,T] \times \mathbb{R})$.

3.21 Global well-posedness

In this section we prove global well-posedness for the SQG equation (3.1.3) with small and localized initial data. We use the wave packet method of Ifrim-Tataru, which is systematically described in [86]. This section is companion to Section 6 in [6], and we shall reproduce a lot of the proofs presented there.

3.21.1 Notation

Consider the linear flow

$$i\partial_t \varphi - A(D)\varphi = 0$$

and the linear operator

$$L = x - tA'(D).$$

In our setting, we have the symbol

$$a(\xi) = -2\xi \log|\xi|$$

and thus

$$A(D) = -2D\log|D|, \qquad L = x + 2t + 2t\log|D|.$$

We define the weighted energy space ($s_0 < 1$ and $s > 3$)

$$\|\varphi\|_X = \|\varphi\|_{\dot H^{s_0}\cap \dot H^s} + \|L\partial_x\varphi\|_{L^2},$$

We also define the pointwise control norm

$$\|\varphi\|_Y = \||D_x|^{1-\delta}\varphi\|_{L^\infty_x} + \||D_x|^{\frac{1}{2}+\delta}\varphi_x\|_{L^\infty_x}.$$

We partition the frequency space into dyadic intervals I_λ localized at dyadic frequencies $\lambda \in 2^{\mathbb{Z}}$, and consider the associated partition of velocities

$$J_\lambda = a'(I_\lambda)$$

which form a covering of the real line, and have equal lengths. To these intervals J_λ we select reference points $v_\lambda \in J_\lambda$, and consider an associated spatial partition of unity

$$1 = \sum_\lambda \chi_\lambda(x), \qquad \operatorname{supp} \chi_\lambda \subseteq \overline{J_\lambda}, \qquad \chi_\lambda = 1 \text{ on } J_\lambda,$$

where $\overline{J_\lambda}$ is a slight enlargement of J_λ, of comparable length, uniformly in λ.

Lastly, we consider the related spatial intervals, tJ_λ, with reference points $x_\lambda = tv_\lambda \in tJ_\lambda$.

3.21.2 Overview of the proof

We provide a brief overview of the proof.

1. We make the bootstrap assumption for the pointwise bound

$$\|\varphi(t)\|_Y \lesssim C\epsilon \langle t\rangle^{-\frac{1}{2}} \tag{3.21.1}$$

where C is a large constant, in a time interval $t \in [0, T]$ where $T > 1$.

2. The energy estimates for (3.1.3) and the linearized equation will imply

$$\|\varphi(t)\|_X \lesssim \langle t\rangle^{C^2\epsilon^2} \|\varphi(0)\|_X. \tag{3.21.2}$$

3. We aim to improve the bootstrap estimate (3.21.1) to

$$\|\varphi(t)\|_Y \lesssim \epsilon \langle t\rangle^{-\frac{1}{2}}. \tag{3.21.3}$$

We use vector field inequalities to derive bounds of the form

$$\|\varphi(t)\|_Y \lesssim \epsilon \langle t\rangle^{-\frac{1}{2}+C\epsilon^2}, \tag{3.21.4}$$

which is the desired bound but with an extra $t^{C\epsilon^2}$ loss.

4. In order to rectify the extra loss, we use the wave packet testing method. Namely, we define a suitable asymptotic profile γ, which is then shown to be an approximate solution for an ordinary differential equation. This enables us to obtain suitable bounds for the asymptotic profile without the aforementioned loss, which can then be transferred back to the solution φ.

3.21.3 Energy estimates

From Proposition 3.19 and Grönwall's lemma, together with the fact that $\epsilon \ll 1$, we get that

$$\|\varphi(t,x)\|_{H^s_x} \lesssim e^{C\int_0^t C(A(\tau))B(\tau)^2\,d\tau} \|\varphi_0\|_{H^s_x}.$$

Let $u = L\partial_x\varphi + t\int F(\delta^y\varphi)|\delta|^y\varphi_x\,dy$, which satisfies the linearized equation with error $\int F'(\delta^y\varphi)\delta^y\varphi|\delta|^y\varphi_x\,dy$, which is clearly balanced. From Proposition 3.16, along with Grönwall's lemma and the fact that $\epsilon \ll 1$, we have

$$\|u(t,x)\|_{L^2_x} \lesssim e^{C\int_0^t C(A(\tau))B(\tau)^2\,d\tau} \|u_0\|_{L^2_x}.$$

Along with the bootstrap assumptions, these readily imply that

$$\|\varphi\|_X \lesssim \|\varphi(t)\|_{H^s_x} + \|u(t)\|_{L^2_x} \lesssim \epsilon e^{C^2\epsilon^2\int_0^t \langle s\rangle^{-1}\,ds} \lesssim \epsilon\langle t\rangle^{C^2\epsilon^2}. \tag{3.21.5}$$

3.21.4 Vector field bounds

Proposition 2.1 from [86] implies that

$$\|\varphi_\lambda\|_{L^\infty_x}^2 \lesssim \frac{1}{t}(\|\varphi_\lambda\|_{L^2_x}\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}^2).$$

When $\lambda \le 1$,

$$\|\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-(1-\delta-\delta_1)}(\|\lambda^{2-2\delta-2\delta_1}\varphi_\lambda\|_{L^2_x}^{1/2}\|L\partial_x\varphi_\lambda\|_{L^2_x}^{1/2} + \|\lambda^{1-\delta-\delta_1}\varphi_\lambda\|_{L^2_x}) \lesssim \frac{1}{\sqrt{t}}\lambda^{-(1-\delta-\delta_1)}\|\varphi\|_X$$

and when $\lambda > 1$,

$$\|\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-\frac{3}{2}-2\delta}(\|\lambda^{3+4\delta}\varphi_\lambda\|_{L^2_x}^{1/2}\|L\partial_x\varphi_\lambda\|_{L^2_x}^{1/2} + \|\lambda^{\frac{3}{2}+2\delta}\varphi_\lambda\|_{L^2_x}) \lesssim \frac{1}{\sqrt{t}}\lambda^{-\left(\frac{3}{2}+2\delta\right)}\|\varphi\|_X.$$

By dyadic summation and Bernstein's inequality, we deduce the bound

$$\|\varphi\|_Y = \|\langle D_x\rangle^{\frac{1}{2}+2\delta}|D_x|^{1-\delta}\varphi\|_{L^\infty_x} \lesssim \frac{\|\varphi\|_X}{\sqrt{t}}. \tag{3.21.6}$$

By the localized dispersive estimate [86, Proposition 5.1],

$$|\varphi_\lambda(x)|^2 \lesssim \frac{1}{|x-x_\lambda|t\frac{1}{\lambda}}(\|L\varphi_\lambda\|_{L^2_x} + \lambda^{-1}\|\varphi_\lambda\|_{L^2_x})^2,$$

which implies that

$$\|(1-\chi_\lambda)\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{-1/2}}{t}(\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{-1/2}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \tag{3.21.7}$$

To end this section we record the following elliptic bounds:

Lemma 3.22. We have

$$\||D_x|^{\frac{1}{2}+\delta}\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x} \lesssim \frac{\lambda^{1+\delta}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \tag{3.22.1}$$

$$\||D_x|^{1-\delta}((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{1}{2}-\delta}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}), \tag{3.22.2}$$

and

$$\|(1-\chi_\lambda)\varphi_\lambda\|_{L^2_x} \lesssim \frac{\lambda^{-1}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}), \tag{3.22.3}$$

Moreover, the difference quotient satisfies the bounds

$$\|(1-\chi_\lambda)\delta^y\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{1/2}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}),$$

and

$$\|(1-\chi_\lambda)\delta^y\varphi_\lambda\|_{L^2_x} \lesssim \frac{(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x})}{t}.$$

Proof. We use the bounds

$$|\partial_x(\chi_\lambda(x/t)| \lesssim t^{-1}.$$

From 3.21.7 applied for $\partial_x\varphi$,

$$\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x} \lesssim \frac{1}{t}\|\chi'_\lambda\varphi_\lambda\|_{L^\infty_x} + \|(1-\chi_\lambda)\partial_x\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{1/2}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}).$$

The first two bounds immediately follow from 3.21.7, and the L^2 elliptic estimate similarly follows from [86, Proposition 5.1].

For the bounds involving the difference quotient, from 3.21.7 applied for $\delta^y\varphi$, we have

$$\begin{aligned}
\|(1-\chi_\lambda)\delta^y\varphi_\lambda\|_{L^\infty_x} &\lesssim \frac{\lambda^{1/2}}{t}(\|L\delta^y\varphi_\lambda\|_{L^2_x} + \lambda^{-1}\|\delta^y\varphi_\lambda\|_{L^2_x}) \\
&\lesssim \frac{\lambda^{1/2}}{t}(\|\delta^y(L\varphi_\lambda)\|_{L^2_x} + \|\varphi_\lambda(x+y)\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \\
&\lesssim \frac{\lambda^{1/2}}{t}(\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \\
&\lesssim \frac{\lambda^{1/2}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x})
\end{aligned}$$

The other bound is proved similarly. □

3.22.1 Wave packets

We construct wave packets as follows. Given the dispersion relation $a(\xi)$, the group velocity v satisfies

$$v = a'(\xi) = -2 - 2\log|\xi|,$$

so we denote

$$\xi_v = -e^{-1-\frac{v}{2}}.$$

Then we define the linear wave packet $\mathbf{u}^v$ associated with velocity v by

$$\mathbf{u}^v = a''(\xi_v)^{-\frac{1}{2}}\chi(y)e^{it\phi(x/t)}, \qquad y = \frac{x-vt}{t^{\frac{1}{2}}a''(\xi_v)^{\frac{1}{2}}},$$

where the phase ϕ is given by

$$\phi(v) = v\xi_v - a(\xi_v),$$

and χ is a unit bump function, such that $\int \chi(y)\,dy = 1$.

We remark that we will typically apply frequency localizations of the form $\mathbf{u}^v_\lambda = P_\lambda\mathbf{u}^v$ with $v \in J_\lambda$.

We observe that since

$$\partial_v(|\xi_v|^{\frac{1}{2}}) = -\frac{1}{4}|\xi_v|^{\frac{1}{2}}, \qquad \partial_v(a''(\xi_v)^{-\frac{1}{2}}) = -\frac{1}{4}a''(\xi_v)^{-\frac{1}{2}},$$

we may write

$$\partial_v \mathbf{u}^v = -\tilde{L}\mathbf{u}^v + \mathbf{u}^{v,II} = t^{\frac{1}{2}}a''(\xi_v)^{-\frac{1}{2}}\mathbf{u}^v + \mathbf{u}^{v,II} \tag{3.22.4}$$

where

$$\tilde{L} = t(\partial_x - i\phi'(x/t))$$

and $\mathbf{u}^{v,II}$ has a similar wave packet form. We also recall from [86, Lemmas 4.4, 5.10] the sense in which $\mathbf{u}^v$ is a good approximate solution:

Lemma 3.23. The wave packet $\mathbf{u}^v$ solves an equation of the form

$$(i\partial_t - A(D))\mathbf{u}^v = t^{-\frac{3}{2}}(L\mathbf{u}^{v,I} + \mathbf{r}^v)$$

where $\mathbf{u}^{v,I}, \mathbf{r}^v$ have wave packet form,

$$\mathbf{u}^{v,I} \approx a''(\xi_v)^{-\frac{1}{2}}\mathbf{u}^v, \qquad \mathbf{r}^v \approx \xi_v^{-1}a''(\xi_v)^{-\frac{1}{2}}\mathbf{u}^v.$$

The asymptotic profile at frequency λ is meaningful when the associated spatial region tJ_λ dominates the wave packet scale at frequency λ:

$$\delta x \approx t^{\frac{1}{2}}a''(\lambda)^{\frac{1}{2}} \lesssim |tJ_\lambda| \approx t\lambda a''(\lambda).$$

This corresponds to

$$t \gtrsim \lambda^{-2}a''(\lambda)^{-1} \approx \lambda^{-1}.$$

Accordingly we define

$$\mathscr{D} = \{(t, v) \in \mathbb{R}^+ \times \mathbb{R} : v \in J_\lambda,\ t \gtrsim \lambda^{-1}\}.$$

3.23.1 Wave packet testing

In this section we establish estimates on the asymptotic profile function

$$\gamma^\lambda(t, v) := \langle \varphi, \mathbf{u}^v_\lambda \rangle_{L^2_x} = \langle \varphi_\lambda, \mathbf{u}^v \rangle_{L^2_x}.$$

We will see that γ^λ essentially has support $v \in J_\lambda$.

We will also use the following crude bounds involving the higher regularity of γ^λ:

Lemma 3.24 (Lemma 6.3, [6])**.** We have

$$\|\chi_\lambda \partial_v^n \gamma^\lambda\|_{L^\infty} \lesssim t^{\frac{1}{2}}(1+t^{\frac{1}{2}}\lambda^{\frac{1}{2}})^n \|\varphi_\lambda\|_{L^\infty_x},$$
$$\|\chi_\lambda \partial_v^n \gamma^\lambda\|_{L^2} \lesssim (t\lambda)^{\frac{1}{4}}(1+t^{\frac{1}{2}}\lambda^{\frac{1}{2}})^n \|\varphi_\lambda\|_{L^2_x},$$

and

$$\|\chi_\lambda \partial_v \gamma^\lambda\|_{L^\infty} \lesssim t^{\frac{1}{4}}\lambda^{-\frac{3}{4}}(1+\lambda^{-1})\|\varphi\|_X + t^{\frac{1}{2}}\|\varphi_\lambda\|_{L^\infty_x}.$$

Proof. Using the second form of $\partial_v \mathbf{u}^v$ in (3.22.4), we have

$$|\chi_\lambda \partial_v \gamma^\lambda| = |\chi_\lambda \langle \varphi_\lambda, \partial_v \mathbf{u}^v\rangle| \lesssim t^{\frac{1}{2}}(t^{\frac{1}{2}}\lambda^{\frac{1}{2}}+1)\|\varphi_\lambda\|_{L^\infty_x}$$

where the $t^{\frac{1}{2}}$ loss in front arises from the L^1 norm of the wave packet. Higher derivatives are obtained similarly, along with the L^2 estimates.

For the last bound, we use the first form of $\partial_v \mathbf{u}^v$ in (3.22.4). The contribution from the wave packet $\mathbf{u}^{v,II}$ is easily estimated as above. For the remaining bound, Lemma 2.3 from [86] implies that

$$|\langle \varphi_\lambda, \tilde{L}\mathbf{u}^v\rangle| \lesssim (t\lambda)^{\frac{1}{4}}\|\tilde{L}\varphi_\lambda\|_{L^2_x} \lesssim t^{\frac{1}{4}}\lambda^{-\frac{3}{4}}\|\varphi_\lambda\|_X,$$

which finishes the proof. □

Approximate profile

We recall from [86] that γ^λ provides a good approximation for the profile of φ. In our setting, we will also need to compare the profile with the differentiated flow $\partial_x \varphi$. Define

$$r^\lambda(t,x) = \chi_\lambda(x/t)\varphi_\lambda(t,x) - t^{-\frac{1}{2}}\chi_\lambda(x/t)\gamma^\lambda(t,x/t)e^{-it\phi(x/t)}.$$

Lemma 3.25 (Lemma 6.4, [6])**.** Let $t \geq 1$. Then we have

$$\|\chi_\lambda(x/t) r^\lambda\|_{L^\infty_x} \lesssim t^{-\frac{3}{4}}\lambda^{-\frac{1}{4}}\|\tilde{L}\varphi_\lambda\|_{L^2_x},$$
$$\|\chi_\lambda(x/t)\partial_v r^\lambda\|_{L^\infty_x} \lesssim t^{\frac{1}{4}}\lambda^{-\frac{1}{4}}\|\tilde{L}\partial_x\varphi_\lambda\|_{L^2_x} + (1+t^{\frac{1}{2}}\lambda^{\frac{1}{2}})\|\varphi_\lambda\|_{L^\infty}.$$

Proof. The first estimate may be obtained from the proof of [86, Proposition 4.7]. For the latter, we use the first representation in (3.22.4) to write

$$e^{it\phi(v)}\partial_v(\gamma(t,v)e^{-it(\phi(v)}) = t\langle \partial_x\varphi_\lambda, \mathbf{u}^v\rangle + \langle \varphi_\lambda, it(\phi'(\cdot/t)-\phi'(v))\mathbf{u}^v\rangle + \langle \varphi_\lambda, \mathbf{u}^{v,II}\rangle. \quad (3.25.1)$$

To address the first term, we see that we may apply the undifferentiated estimate with $\partial_x\varphi_\lambda$ in place of φ_λ. Precisely, we may apply the first estimate on

$$\partial_x\varphi_\lambda(t,x) - t^{-\frac{1}{2}}\langle \partial_x\varphi_\lambda, \mathbf{u}^{x/t}\rangle e^{-it\phi(x/t)}.$$

We estimate the third term of (3.25.1) via

$$t^{-\frac{1}{2}}|\langle \varphi_\lambda, \mathbf{u}^{v,II}\rangle| \lesssim \|\varphi_\lambda\|_{L^\infty}.$$

It remains to estimate the middle term,

$$t^{-\frac{1}{2}}|\chi_\lambda(v)\langle \varphi_\lambda, it(\phi'(\cdot/t)-\phi'(v))\mathbf{u}^v\rangle| \lesssim |\phi''(\lambda)|\cdot t^{\frac{1}{2}}a''(\lambda)^{\frac{1}{2}}\cdot\|\varphi_\lambda\|_{L^\infty} \lesssim t^{\frac{1}{2}}\lambda^{\frac{1}{2}}\|\varphi_\lambda\|_{L^\infty}$$

□

We also observe that on the wave packet scale, we may replace $\gamma(t,v)$ with $\gamma(t,x/t)$ up to acceptable errors. Denote

$$\beta_v^\lambda(t,x) = t^{-1/2}\chi_\lambda(x/t)(\gamma(t,v)-\gamma(t,x/t))e^{it\phi(x/t)},$$

Lemma 3.26 (Lemma 6.5, [6])**.** Let $v \in J_\lambda$, and $(t,v) \in \mathscr{D}$. Then, for every $y \neq 0$ and x such that $|x-vt| \lesssim \delta x = t^{1/2}\lambda^{-1/2}$, we have the bound

$$|\delta^y \beta_v| \lesssim t^{-3/4}\lambda^{-1/4}\|\varphi\|_X$$

Proof. We have

$$\delta^y\beta_v = -t^{-1/2}\delta^y(\gamma(t,\cdot/t))\chi_\lambda((x+y)/t)e^{it\phi((x+y)/t)} + t^{-1/2}(\gamma(t,v)-\gamma(t,x/t))\delta^y(\chi_\lambda(\cdot/t)e^{it\phi(\cdot/t)}).$$

The Mean Value Theorem ensures that

$$|\delta^y(\gamma(t,\cdot/t))| \lesssim t^{-1}\|\partial_v\gamma\|_{L^\infty},$$

and that

$$\begin{aligned}
|\delta^y\beta_v| &\lesssim t^{-1/2}(t^{-1}\|\partial_v\gamma\|_{L^\infty} + t^{-1/2}\lambda^{-1/2}\|\partial_v\gamma\|_{L^\infty}(t^{-1}+\lambda)) \\
&\lesssim t^{-1}\|\partial_v\gamma\|_{L^\infty}(t^{-1/2}+\lambda^{1/2}) \lesssim t^{-1}\lambda^{1/2}(\lambda^{-3/4}t^{1/4}\|\varphi\|_X + \|\varphi_\lambda\|_{L_x^\infty}t^{1/2}) \\
&\lesssim t^{-3/4}\lambda^{-1/4}\|\varphi\|_X + t^{-1/2}\lambda^{1/2}\|\varphi_\lambda\|_{L_x^\infty} \lesssim t^{-3/4}\lambda^{-1/4}\|\varphi\|_X + t^{-1}\lambda^{-1/4}\|\varphi\|_X \\
&\lesssim t^{-3/4}\lambda^{-1/4}\|\varphi\|_X
\end{aligned}$$

□

3.26.1 Bounds for Q

Write, slightly abusing notation,

$$Q(\varphi) = Q(\varphi,\overline{\varphi},\varphi) := \frac{1}{3}\int \operatorname{sgn}(y)\cdot|\delta^y\varphi|^2\delta^y\varphi\,dy.$$

We recall from [6] the following lemma:

Lemma 3.27 (See Lemma 6.6,[6])**.** For $0 < \delta \ll 1$, we have the difference estimates

$$\begin{aligned}\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_x + L^{1/\delta}_x} &\lesssim \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1, \varphi_2)\|^2_{L^\infty_x} \\ &\quad + \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1, \varphi_2)\|_{L_x^{\frac{1}{2\delta}}} \|(\varphi_1, \varphi_2)\|_{L^\infty_x},\end{aligned}$$

$$\begin{aligned}\|Q(\varphi_1) - Q(\varphi_2)\|_{L^2_x} &\lesssim \|\partial_x(\varphi_1, \varphi_2)\|_{L^2_x} \|\partial_x(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x} \\ &\quad + \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{L^2_x} \|(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},\end{aligned}$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_x + L^{1/\delta}_x} \lesssim (\|\langle D_x\rangle^\delta(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1, \varphi_2)\|_{L^\infty_x + L_x^{\frac{1}{2\delta}}} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^2_x} \lesssim (\|\langle D_x\rangle^\delta(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1, \varphi_2)\|_{L^2_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}.$$

Proof. We are only going to prove the first two estimates, as the proofs of the other two are similar. Write

$$Q(\varphi_1) - Q(\varphi_2) = \int_{|y|\le 1} + \int_{|y|>1}$$

where the integrand may be written

$$\begin{aligned}&\operatorname{sgn}(y)(|\delta^y\varphi_1|^2\delta^y\overline{\varphi_1} - |\delta^y\varphi_2|^2\delta^y\overline{\varphi_2}) \\ &\qquad = \operatorname{sgn}(y)(\delta^y(\varphi_1 - \varphi_2)(|\delta^y\varphi_1|^2 + |\delta^y\varphi_2|^2) + \delta^y(\overline{\varphi_1} - \overline{\varphi_2})\delta^y\varphi_1\delta^y\varphi_2).\end{aligned}$$

The first integral contributes to the two estimates respectively,

$$\left|\int_{|y|\le 1}\right| \lesssim \|\partial_x(\varphi_1, \varphi_2)\|^2_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}$$

and

$$\left\|\int_{|y|\le 1}\right\|_{L^2} \lesssim \|\partial_x(\varphi_1, \varphi_2)\|_{L^2_x} \|\partial_x(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}.$$

For the second, using Sobolev embedding,

$$\begin{aligned}\left\|\int_{|y|>1}\right\|_{L^{1/\delta}} &\lesssim \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{L^{1/\delta}_x} \|(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x} \\ &\lesssim \|\partial_x(\varphi_1, \varphi_2)\|_{L^{1/(2\delta)}_x} \|(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}\end{aligned}$$

and

$$\left\|\int_{|y|>1}\right\|_{L^2} \lesssim \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{L^2_x} \|(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}.$$

$\square$

We will be considering separately the balanced and unbalanced components of Q. Precisely, we denote the diagonal set of frequencies by $\mathscr{D}$ and write

$$\begin{aligned} Q(\varphi,\varphi,\varphi) &= \sum_{(\lambda_1,\lambda_2,\lambda_3,\lambda)\in\mathscr{D}} Q(\varphi_{\lambda_1},\varphi_{\lambda_2},\varphi_{\lambda_3}) + \sum_{(\lambda_1,\lambda_2,\lambda_3,\lambda)\notin\mathscr{D}} Q(\varphi_{\lambda_1},\varphi_{\lambda_2},\varphi_{\lambda_3}) \\ &= Q^{bal}(\varphi,\varphi,\varphi) + Q^{unbal}(\varphi,\varphi,\varphi) = Q^{bal}(\varphi) + Q^{unbal}(\varphi). \end{aligned}$$

The unbalanced portion of Q satisfies the better bound as follows:

Lemma 3.28. Q^{unbal} satisfies the bounds

$$\|\chi_\lambda^1 \partial_x P_\lambda Q^{unbal}(\varphi)\|_{L_x^\infty} \lesssim \lambda^{-\delta} \frac{\|\varphi\|_X^3}{t^2}$$

and

$$\|\chi_\lambda^1 \partial_x P_\lambda Q^{unbal}(\varphi)\|_{L_x^2} \lesssim \lambda^{-\delta} \frac{\|\varphi\|_X^3}{t^{3/2}},$$

where χ_λ^1 is a cut-off widening χ_λ.

Proof. We shall denote

$$I_{\lambda_1,\lambda_2,\lambda_3} = \int_{\mathbb{R}} \operatorname{sgn}(y)\delta^y\varphi_{\lambda_1}\delta^y\varphi_{\lambda_2}\delta^y\varphi_{\lambda_3}\,dy$$

and consider two cases in the frequency sum for $\partial_x P_\lambda Q^{unbal}$.

First we consider the case in which we have two low separated frequencies. We assume without loss of generality that $\lambda_3 = \lambda$ and $\lambda_1 < \lambda_2 \ll \lambda$. In this case, the elliptic estimates will be applied for the factor φ_{λ_1}. Precisely, from Lemma 3.5 and estimates 3.21.6, 3.21.7, and 3.22.3, we get that

$$\begin{aligned} \left\|\chi_\lambda^1 I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} &\lesssim \lambda_1 \frac{\lambda_1^{-1/2}}{t}\|\varphi\|_X(\lambda_2^{1-2\delta}+\lambda_2)\|\varphi_{\lambda_2}\|_{L_x^\infty}\lambda_3^{\delta}\|\varphi_{\lambda_3}\|_{L_x^\infty} \\ &\lesssim \frac{\lambda_1^{1/2}}{t}\|\varphi\|_X\lambda_2^{\delta}(\lambda_2^{1-3\delta}+\lambda_2^{1-\delta})\|\varphi_{\lambda_2}\|_{L_x^\infty}\lambda^{-\frac{3}{2}-2\delta}\lambda^{\frac{3}{2}+3\delta}\|\varphi_\lambda\|_{L_x^\infty} \\ &\lesssim \lambda_1^{1/2}\lambda_2^{\delta}\lambda^{-\frac{3}{2}-2\delta}\frac{\|\varphi\|_X^3}{t^2}. \end{aligned}$$

By using dyadic summation in λ_1 and λ_2, we deduce that

$$\left\|\chi_\lambda^1\partial_x \sum_{\lambda_1<\lambda_2\ll\lambda} I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} \lesssim \lambda^{-\delta}\frac{\|\varphi\|_X^3}{t^2}.$$

Similarly, we deduce that

$$\left\| \chi_\lambda^1 \partial_x \sum_{\lambda_1 < \lambda_2 \ll \lambda} I_{\lambda_1,\lambda_2,\lambda_3} \right\|_{L^2_x} \lesssim \lambda^{-\delta} \frac{\|\varphi\|_X^3}{t^{3/2}}$$

We now analyze the situation in which $\lambda_1, \lambda_2 \gtrsim \lambda$, and λ_1 and λ_2 are comparable and both separated from λ. Thus, we will be able to use λ_1 and λ_2 interchangeably. We replace χ_λ^1 by $\tilde{\chi}_\lambda$, which has double support, and equals 1 on a comparably-sized neighbourhood of the support of χ_λ^1. We write

$$\chi_\lambda^1 \partial_x P_\lambda = \chi_\lambda^1 \partial_x P_\lambda \tilde{\chi}_\lambda + \chi_\lambda^1 \partial_x P_\lambda (1 - \tilde{\chi}_\lambda).$$

For the first term, using Lemma 3.5, along with estimates 3.21.6, 3.21.7, 3.22.3, we get the bounds

$$\begin{aligned} \left\| \chi_\lambda^1 P_\lambda \tilde{\chi}_\lambda I_{\lambda_1,\lambda_2,\lambda_3} \right\|_{L^\infty_x} &\lesssim \lambda_2^{1/2+\delta} \frac{\lambda_3^{1-2\delta} + \lambda_3}{t} \|\varphi\|_X \|\varphi_{\lambda_2}\|_{L^\infty_x} \|\varphi_{\lambda_3}\|_{L^\infty_x} \\ &\lesssim \lambda_2^{-1-\delta/2} \lambda_3^{\delta/2} \frac{\|\varphi\|_X}{t} (\lambda_3^{1-5\delta/2} + \lambda_3^{1-\delta/2}) \|\varphi_{\lambda_3}\|_{L^\infty_x} \lambda_2^{3/2+3\delta/2} \|\varphi_{\lambda_2}\|_{L^\infty_x} \\ &\lesssim \lambda_2^{-1-\delta/2} \lambda_3^{\delta/2} \frac{\|\varphi\|_X^3}{t^2} \end{aligned}$$

and

$$\begin{aligned} \left\| \chi_\lambda^1 P_\lambda \tilde{\chi}_\lambda I_{\lambda_1,\lambda_2,\lambda_3} \right\|_{L^2_x} &\lesssim \lambda_2^{\delta} \frac{\lambda_3^{1-2\delta} + \lambda_3}{t} \|\varphi\|_X \|\varphi_{\lambda_2}\|_{L^\infty_x} \|\varphi_{\lambda_3}\|_{L^\infty_x} \\ &\lesssim \lambda_2^{-1-3\delta/2} \lambda_3^{\delta/2} \frac{\|\varphi\|_X}{t} (\lambda_3^{1-5\delta/2} + \lambda_3^{1-\delta/2}) \|\varphi_{\lambda_3}\|_{L^\infty_x} \lambda_2^{1+5\delta/2} \|\varphi_{\lambda_2}\|_{L^\infty_x} \\ &\lesssim \lambda_2^{-1-3\delta/2} \lambda_3^{\delta/2} \frac{\|\varphi\|_X^3}{t^2}. \end{aligned}$$

By using dyadic summation in λ_1, λ_2, and λ_3 (and by using the fact that λ_1 and λ_2 are close), we deduce the bound

$$\left\| \chi_\lambda^1 \partial_x P_\lambda \tilde{\chi}_\lambda \sum_{\lambda_3 \lesssim \lambda_2, \lambda_1 \simeq \lambda_2 \gtrsim \lambda} I_{\lambda_1,\lambda_2,\lambda_3} \right\|_{L^\infty_x \cap L^2_x} \lesssim \lambda^{-\delta} \frac{1}{t^2} \|\varphi\|_X^3.$$

We look at the second term. For every N, we know that

$$\|\chi_\lambda^1 \partial_x P_\lambda (1 - \tilde{\chi}_\lambda)\|_{L^2 \to L^2}, \|\chi_\lambda^1 \partial_x P_\lambda (1 - \tilde{\chi}_\lambda)\|_{L^\infty \to L^\infty} \lesssim \frac{\lambda^{1-N}}{t^N}$$

We take $N = \frac{3}{2}$. By carrying out a similar analysis as above, along with Lemma 3.5 and dyadic summation, we deduce that the contributions corresponding to these terms are also acceptable. □

Lemma 3.29 (Lemma 6.8 [6]). We have

$$\chi_\lambda((x/t))^3 Q(e^{it\phi(x/t)}) = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} q(\phi'(x/t)) + h(\lambda, t),$$

where for every $a \in (0,1)$

$$|h(\lambda,t)| \lesssim \frac{\lambda^3}{t^{2-3a}} + \frac{\lambda^2}{t^{1-a}} + \frac{1}{t^{2a}}$$

Proof. We write

$$e^{-it\phi(x/t)}\delta^y e^{it\phi(x/t)} = \frac{e^{iy\phi'(x/t)}\left(e^{it/2\phi''(c_{x,y}/t)y^2/t^2} - 1\right)}{y} + \frac{e^{iy\phi'(x/t)} - 1}{y} =: a + b,$$

where $c_{x,y}$ is between x and $x+y$. We now use the fact that x/t belongs to the support of χ_λ. We have

$$|\chi_\lambda(x/t)||b| \lesssim \lambda.$$

Moreover, when $|y| \leq t^a$, $|c_{x,y}/t - x/t| \leq |y/t| \leq t^{a-1}$. This implies that $c_{x,y}/t$ belongs to the support of the enlarged cut-off χ^1_λ, hence $\phi''(c_{x,y}/t) \simeq \lambda$. We note the bound

$$|\chi_\lambda(x/t)||a| \lesssim |\chi_\lambda(x/t)||y/(2t)\phi''(c_{x,y}/t)| \left| \frac{e^{\pm i\phi''(c_{x,y}/t)y^2/(2t)} - 1}{\phi''(c_{x,y}/t)y^2/(2t)} \right| \lesssim \lambda t^{a-1}$$

Thus, we have the bounds

$$\begin{aligned} |\chi_\lambda(x/t)||a| &\lesssim \lambda t^{a-1} \\ |\chi_\lambda(x/t)||b| &\lesssim \lambda \end{aligned} \tag{3.29.1}$$

We also note the cruder bounds

$$|\chi_\lambda(x/t)||a| + |\chi_\lambda(x/t)||b| \lesssim \frac{1}{|y|} \tag{3.29.2}$$

We write

$$\begin{aligned} (\chi_\lambda(x/t))^3 Q(e^{it\phi(x/t)}) &= (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int |b|^2 \, b \, dy \\ &+ (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int a^2\overline{a} + a^2\overline{b} + 2|a|^2 b + 2a|b|^2 + b^2\overline{a} \, dy := T_1 + T_2 \end{aligned}$$

We note that

$$T_1 = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int |b|^2 \, b \, dy = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} q(\phi'(x/t)),$$

so we only need to analyze T_2.

We first bound the contribution over the region $|y| \leq t^a$, which we shall denote by T_2^1. We denote the contribution over the region $|y| > t^a$ by T_2^2. We have

$$\begin{aligned} T_2^1 &= (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int_{|y|\leq t^a} a^2\overline{a} + a^2\overline{b} + 2|a|^2 b \, dy \\ &+ (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int_{|y|\leq t^a} 2a|b|^2 + b^2\overline{a} \, dy := T_{2a} + T_{2b}, \end{aligned}$$

3.29.1 implies that

$$\begin{aligned} |T_{2a}| &\lesssim |\chi_\lambda(x/t)|^3 \int_{|y|\leq t^a} |a|^3 + |a|^2|b| \, dy \lesssim \int_{|y|\leq t^a} \frac{\lambda^2}{t^{2-2a}} \left(\frac{\lambda}{t^{1-a}} + \lambda \right) \lesssim \int_{|y|\leq t^a} \frac{\lambda^3}{t^{2-2a}} \, dy \\ &\lesssim t^a \frac{\lambda^3}{t^{2-2a}} \lesssim \frac{\lambda^3}{t^{2-3a}} \end{aligned}$$

3.29.1 and 3.29.2 imply the bound

$$\begin{aligned} |\chi_\lambda(x/t)|^3 |b|^2 |a| &= |\chi_\lambda(x/t)|^3 \, |b|^2 \, |y/(2t)\phi''(c_{x,y}/t)| \left| \frac{e^{\pm i\phi''(c_{x,y}/t)y^2/(2t)} - 1}{\phi''(c_{x,y}/t)y^2/(2t)} \right| \\ &\lesssim \lambda \frac{1}{|y|} |\chi_\lambda(x/t)| |y/(2t)\phi''(c_{x,y}/t)| \lesssim \frac{\lambda^2}{t} \end{aligned}$$

It follows that T_{2b} satisfies the bound

$$|T_{2b}| \lesssim |\chi_\lambda(x/t)|^3 \int_{|y|\leq t^a} |b|^2 \, |a| \, dy \lesssim \int_{|y|\leq t^a} \frac{\lambda^2}{t} \, dy \lesssim \frac{\lambda^2}{t^{1-a}}$$

For T_2^2, 3.29.2 implies that

$$|T_2^2| \lesssim \int_{|y|>t^a} \frac{1}{|y|^3} \, dy \lesssim \frac{1}{t^{2a}}.$$

□

This result can be viewed as a semiclassical expansion of the cubic form Q applied to the wave packet phase correction, and will be useful in deriving an asymptotic ordinary differential equation for the profile γ^λ.

3.29.1 The asymptotic equation for γ

Here we record the error bounds for the asymptotic equation for γ. The proof follows precisely that of Proposition 6.9 in [6]. The change in exponents corresponds to the one in Lemma 3.28.

Proposition 3.30. Let $v \in J_\lambda$. Under the assumption $(t, v) \in \mathscr{D}$, we have

$$\dot{\gamma}(t, v) = iq(\xi_v)\xi_v t^{-1}\gamma(t, v)|\gamma(t, v)|^2 + f(t, v),$$

where

$$|f(t, v)| \lesssim \lambda^{-\delta} g(\lambda) t^{-1-\delta+C\epsilon^2}\epsilon,$$

where $g(\lambda)$ is a finite sum of powers of λ, and

$$\|f(t, v)\|_{L^2_v(J_\lambda)} \lesssim \lambda^{-\delta}(1 + \lambda^{-\frac{1}{2}})t^{-1-\delta+C\epsilon^2}\epsilon.$$

Proof. We have

$$\dot{\gamma}(t, v) = \langle \dot{\varphi}, \mathbf{u}^v_\lambda \rangle + \left\langle \varphi, \dot{\mathbf{u}^v_\lambda} \right\rangle = \langle P_\lambda A_\varphi \varphi, \mathbf{u}^v \rangle + i\langle \varphi_\lambda, (i\partial_t - A(D))\mathbf{u}^v \rangle := I_1 + I_2.$$

We first analyze I_2. We use Lemma 3.23 to write

$$(i\partial_t - A(D))\mathbf{u}^v = t^{-\frac{3}{2}}(L\mathbf{u}^{v,I} + \mathbf{r}^v)$$

$$\begin{aligned}
|\langle \varphi_\lambda, (i\partial_t - A(D))\mathbf{u}^v \rangle| &\lesssim t^{-\frac{3}{2}}(\|L\varphi_\lambda\|_{L^2_x} \cdot \lambda^{1/2}\lambda^{1/4}t^{1/4} + \|\varphi_\lambda\|_{L^2_x} \cdot \lambda^{-1/2}\lambda^{1/4}t^{1/4}) \\
&\lesssim \lambda^{-1/4}t^{-5/4}(1 + \lambda^{-1})\|\varphi\|_X \\
&\lesssim \lambda^{-1/4}t^{-5/4}(1 + \lambda^{-1})\epsilon t^{C\epsilon^2}
\end{aligned}$$

and

$$\begin{aligned}
\|\chi_\lambda \langle \varphi_\lambda, (i\partial_t - A(D))\mathbf{u}^v \rangle\|_{L^2_v} &\lesssim t^{-3/2}\left(\|L\varphi_\lambda\|_{L^2_x}\lambda^{1/2} + \|\varphi_\lambda\|_{L^2_x}\lambda^{-1/2}\right) \\
&\lesssim \lambda^{-1/4}t^{-5/4}t^{-1/4}\lambda^{-1/4}(1 + \lambda^{-1})\|\varphi\|_X \\
&\lesssim \lambda^{-1/4}t^{-5/4}(1 + \lambda^{-1})\epsilon t^{C\epsilon^2} \lesssim \lambda^{-1/4}(1 + \lambda^{-1})t^{-6/5}\epsilon t^{C\epsilon^2}
\end{aligned}$$

(we have used the condition $(t, v) \in \mathscr{D}$.)

In the remaining part of this section we shall analyze the term I_1. We first exchange F for its principal quadratic term, expanding

$$F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2 = \int_0^1 \frac{(1-h)^2}{2}(\delta^y\varphi)^3 F'''(h\delta^y\varphi)\, dh.$$

From Moser's estimate (the nonlinear version, as well as the one for products), Lemma 3.5, and Sobolev embedding, we get that

$$\begin{aligned}
&\lambda^{\frac{1}{4}} \left\| P_\lambda \int (\delta^y \varphi)^3 F'''(t\delta^y\varphi)|\delta|^y \varphi_x \, dy \right\|_{L^\infty_x} \\
&\lesssim \int \| |D_x|^{1/2} ((\delta^y\varphi)^3 |\delta|^y \varphi_x F'''(t\delta^y\varphi)) \|_{L^4_x} \, dy \\
&\lesssim \|\varphi_x\|_{L^\infty_x} \| |D_x|^{\frac{1}{2}} \varphi_x \|_{L^4_x} (\| |D_x|^{1-\delta}\varphi\|_{L^\infty_x} \|\varphi_x\|^2_{L^\infty_x} + \|\varphi_x\|^2_{L^\infty_x} \|\varphi_{xx}\|_{L^\infty_x}) \\
&+ \|\varphi_x\|_{L^\infty_x} \| |D_x|^{\frac{1}{2}-\delta} \langle D_x\rangle^\delta \varphi_x \|_{L^4_x} (\|\varphi_x\|^3_{L^\infty_x} + \|\varphi_x\|^2_{L^\infty_x} \| |D_x|^{1-\delta}\varphi\|_{L^\infty_x}) \\
&\lesssim \frac{1}{t^{5/4}} \epsilon^5 \langle t\rangle^{C\epsilon^2}.
\end{aligned}$$

We have also used Sobolev embedding and the classical Moser estimate, keeping in mind $F'''(0) = 0$. Similarly,

$$\left\| \int_{\mathbb{R}} P_\lambda ((F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2)|\delta|^y \varphi_x) \, dy \right\|_{L^2_x} \lesssim \lambda^{-1/4} \frac{1}{t^{3/2}} \epsilon^5 \langle t\rangle^{C\epsilon^2}.$$

By Hölder's inequality and Young's inequality respectively,

$$\begin{aligned}
\left| \left\langle P_\lambda \int_{\mathbb{R}} \left(F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2 \right) |\delta|^y \varphi_x \, dy, \varphi_v \right\rangle \right| &\lesssim \lambda^{-1/4} \frac{1}{t^{3/2}} \epsilon^5 \langle t\rangle^{C\epsilon^2}, \\
\left\| \left\langle P_\lambda \int_{\mathbb{R}} \left(F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2 \right) |\delta|^y \varphi_x \, dy, \varphi_v \right\rangle \right\|_{L^2_v(J_\lambda)} &\lesssim \lambda^{-1/4} \frac{1}{t^{3/2}} \epsilon^5 \langle t\rangle^{C\epsilon^2}.
\end{aligned}$$

We are left to estimate

$$\begin{aligned}
\left\langle P_\lambda \int_{\mathbb{R}} (\delta^y\varphi)^2 |\delta|^y \varphi_x \, dy, \mathbf{u}^v \right\rangle = \left\langle \partial_x P_\lambda Q^{\text{bal}}(\varphi), \mathbf{u}^v \right\rangle + \left\langle \chi^1_\lambda \partial_x P_\lambda Q^{\text{unbal}}(\varphi), \mathbf{u}^v \right\rangle \\
+ \left\langle (1-\chi^1_\lambda) \partial_x P_\lambda Q^{\text{unbal}}(\varphi), \mathbf{u}^v \right\rangle,
\end{aligned}$$

where χ^1_λ be a cut-off function enlarging χ_λ. Due to the fact that $\mathbf{u}^v$ is supported in the region $\left| \frac{x}{t} - v \right| \lesssim \lambda^{-1/2} t^{-1/2}$, the condition $(t, v) \in \mathscr{D}$ will imply that the third term is identically zero, while Lemma 3.28 implies that the second term is an acceptable error. Thus, we only have to analyze

$$\left\langle \partial_x P_\lambda Q^{\text{bal}}(\varphi), \mathbf{u}^v \right\rangle = \left\langle \partial_x P_\lambda Q(\varphi_\lambda), \mathbf{u}^v \right\rangle.$$

Let χ^1 be a cut-off function that is equal to 1 on the support of the wave packet $\mathbf{u}_v$. Let $\tilde{\chi}$ be another cut-off function whose support is slightly larger than the one of χ^1. We write

$$\left\langle \partial_x P_\lambda Q(\varphi_\lambda), \mathbf{u}^v \right\rangle = \left\langle \partial_x P_\lambda \tilde{\chi} Q(\varphi_\lambda), \mathbf{u}^v \right\rangle + \left\langle \chi^1 \partial_x P_\lambda (1-\tilde{\chi}) Q(\varphi_\lambda), \mathbf{u}^v \right\rangle$$

As in the proof of Lemma 3.28, we note that the operator norm bounds

$$\|\chi^1\partial_x P_\lambda(1-\tilde{\chi})\|_{L^\infty\to L^\infty} + \|\chi^1\partial_x P_\lambda(1-\tilde{\chi})\|_{L^2\to L^2} \lesssim \lambda^{1-2N}t^{-N}$$

for every N imply that the second term is acceptable error. This leaves us with the first.

We first replace φ_λ by $\chi_\lambda\varphi_\lambda$. From Lemma 3.27, we have

$$\begin{aligned}
&|\langle \partial_x P_\lambda \tilde{\chi}(Q(\varphi_\lambda) - Q(\chi_\lambda\varphi_\lambda)), \mathbf{u}^v\rangle| \\
&\qquad \lesssim \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\varphi_\lambda)\|^2_{L^\infty_x}\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x}(\|\mathbf{u}^v\|_{L^1_x} + \|\mathbf{u}^v\|_{L_x^{\frac{1}{1-\delta}}}) \\
&\qquad + \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\varphi_\lambda)\|_{L_x^{\frac{1}{2\delta}}}\|(\chi_\lambda\varphi_\lambda, \varphi_\lambda)\|_{L^\infty_x}\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x}(\|\mathbf{u}^v\|_{L^1_x} + \|\mathbf{u}^v\|_{L_x^{\frac{1}{1-\delta}}})
\end{aligned}$$

By interpolation, along with Lemma 3.22 and the condition $(t,v) \in \mathscr{D}$, it follows that the errors are acceptable. The L^2_x-bound is similar.

We now denote

$$\psi(t,x) = t^{-\frac{1}{2}}\chi_\lambda(x/t)\gamma(t,x/t)e^{it\phi(x/t)}$$

and replace $\chi_\lambda\varphi_\lambda$ by ψ. From Lemma 3.27, we have

$$\begin{aligned}
&|\langle \partial_x P_\lambda \tilde{\chi}(Q(\chi_\lambda\varphi_\lambda) - Q(\psi)), \mathbf{u}^v\rangle| \\
&\qquad \lesssim \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\psi)\|^2_{L^\infty_x}\|\partial_x(\chi_\lambda(x/t)r^\lambda)\|_{L^\infty_x}(\|\mathbf{u}^v\|_{L^1_x} + \|\mathbf{u}^v\|_{L_x^{\frac{1}{1-\delta}}}) \\
&\qquad + \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\psi)\|_{L_x^{\frac{1}{2\delta}}}\|(\chi_\lambda\varphi_\lambda, \psi)\|_{L^\infty_x}\|\partial_x(\chi_\lambda(x/t)r^\lambda)\|_{L^\infty_x}(\|\mathbf{u}^v\|_{L^1_x} + \|\mathbf{u}^v\|_{L_x^{\frac{1}{1-\delta}}})
\end{aligned}$$

By interpolation, along with Lemmas 3.25 and 3.24, and the condition $(t,v) \in \mathscr{D}$, it follows that the errors are acceptable. The L^2_x-bound is similar.

We now denote

$$\theta(t,x) = t^{-\frac{1}{2}}\chi_\lambda(x/t)\gamma(t,v)e^{it\phi(x/t)}$$

and replace ψ by θ. We evaluate

$$\langle \partial_x P_\lambda \tilde{\chi}(Q(\psi) - Q(\theta)), \mathbf{u}^v\rangle$$

We have

$$|\tilde{\chi}(Q(\psi) - Q(\theta))| \lesssim \left|\tilde{\chi}\left(\frac{x - vt}{\sqrt{|ta''(\xi_v)|}}\right)\right| \int (|\delta^y\psi|^2 + |\delta^y\theta|^2)|\delta^y\beta_v^\lambda(x)|\,dy$$

The support condition of $\tilde{\chi}$ implies that x is in the region $|x - vt| \lesssim \delta x = t^{1/2}\lambda^{-1/2}$. From Lemma 3.26 we now get that

$$|\tilde{\chi}(Q(\psi) - Q(\theta))| \lesssim t^{-3/4}\lambda^{-1/4}\|\varphi\|_X \int (|\delta^y\psi|^2 + |\delta^y\theta|^2)\,dy$$

Bernstein's inequality, and Lemma 3.5, and Sobolev embedding, imply that

$$\begin{aligned}
|\tilde{\chi}(Q(\psi) - Q(\theta))| &\lesssim t^{-3/4}\lambda^{3/4}\|\varphi\|_X(\|\psi_x\|^2_{L^\infty_x} + \|\theta_x\|^2_{L^\infty})\|\mathbf{u}^v\|_{L^1_x} \\
&\quad + t^{-3/4}\lambda^{3/4}\|\varphi\|_X(\|\psi\|_{L^\infty_x}\||D_x|^{1-\delta}\psi\|_{L^{\frac{1}{\delta}}_x} + \|\theta\|_{L^\infty_x}\||D_x|^{1-\delta}\theta\|_{L^{\frac{1}{\delta}}_x})\|\mathbf{u}^v\|_{L_x^{\frac{1}{1-\delta}}} \\
&\lesssim t^{-3/4}\lambda^{3/4}\|\varphi\|_X(\|\psi_x\|^2_{L^\infty_x} + \|\theta_x\|^2_{L^\infty})\|\mathbf{u}^v\|_{L^1_x} \\
&\quad + t^{-3/4}\lambda^{3/4}\|\varphi\|_X(\|\psi\|_{L^\infty_x}\|\psi_x\|_{L_x^{\frac{1}{2\delta}}} + \|\theta\|_{L^\infty_x}\|\theta_x\|_{L_x^{\frac{1}{2\delta}}})\|\mathbf{u}^v\|_{L_x^{\frac{1}{1-\delta}}}
\end{aligned}$$

From Lemma 3.24 along with the condition $(t,v) \in \mathscr{D}$, it follows that this error is acceptable. The L^2_x-bound is similar.

We are left to analyze

$$t^{-3/2}\gamma(t,v)|\gamma(t,v)|^2\left\langle \partial_x P_\lambda Q(\chi_\lambda e^{it\phi(x/t)}), \mathbf{u}^v \right\rangle.$$

Since by Lemma 3.24,

$$\|t^{-3/2}\gamma(t,v)|\gamma(t,v)|^2\|_{L^\infty_v(J_\lambda)} \lesssim \|\varphi_\lambda\|^3_{L^\infty_x}, \qquad \|t^{-3/2}\gamma(t,v)|\gamma(t,v)|^2\|_{L^2_v(J_\lambda)} \lesssim t^{-1/2}\|\varphi_\lambda\|^2_{L^\infty_x}\|\varphi_\lambda\|_{L^2_x}$$

it suffices to estimate

$$|\langle \partial_x P_\lambda Q(\chi_\lambda e^{it\phi(x/t)}), \mathbf{u}^v\rangle - t^{\frac{1}{2}}q(\xi_v)\xi_v(\chi_\lambda(v))^3| \lesssim \lambda^{-1/4}t^{3/10+C\epsilon^2}\epsilon.$$

We note that

$$\delta^y(\chi_\lambda e^{\pm it\phi(x/t)}) = \chi_\lambda\delta^y\left(e^{\pm it\phi(x/t)}\right) + \delta^y(\chi_\lambda)e^{\pm it\phi((x+y)/t)}.$$

Lemma 3.5, implies that for every $\delta > 0$ we have

$$\left|\left\langle \partial_x P_\lambda \int \delta^y(\chi_\lambda)e^{it\phi((x+y)/t)}\delta^y(\chi_\lambda e^{-it\phi(x/t)})\delta^y(\chi_\lambda e^{it\phi(x/t)})\, dy, \mathbf{u}^v \right\rangle\right| \lesssim \lambda(t^{\delta-1/2}\lambda + t^{-1/2}\lambda^2).$$

The most problematic contribution is the one that arises from the first term. We have

$$\lambda^3 t^{\delta-1/2}\|\varphi_\lambda\|^3_{L^\infty_x} \lesssim t^{\delta-2}\|\varphi\|^3_X \lesssim t^{-6/5}\epsilon^3 t^{C\epsilon^2}$$

The other term is analogous.

The L^2_v-bound is treated similarly, and so is the case in which one chooses the term $\delta^y(\chi_\lambda)e^{-it\phi((x+y)/t)}$ in the expansion of $\delta^y(\chi_\lambda e^{-it\phi(x/t)})$. This leaves us with

$$\langle \partial_x P_\lambda \left(\chi_\lambda(x/t)^3 Q(e^{it\phi(x/t)})\right), \mathbf{u}^v\rangle$$

Lemma 3.29, implies that we can replace the latter with

$$\langle \partial_x P_\lambda \left(\chi_\lambda(x/t)^3 e^{it\phi(x/t)}(\phi'(x/t))^2 q(1)\right), \mathbf{u}^v\rangle,$$

with error bounded by

$$\lambda t^{1/2}\left(\frac{\lambda^3}{t^{2-3a}}+\frac{\lambda^2}{t^{1-a}}+\frac{1}{t^{2a}}\right)$$

We note that one problematic contribution is the one arising from the last term. We have the bound

$$\frac{\lambda^{5/4}}{t^{2a-1/2}}\|\varphi_\lambda\|^3_{L^\infty_x}\lesssim t^{-2a}\|\varphi\|^3_X$$

By picking $a=\frac{3}{5}$, we deduce that this contribution is acceptable. The only other problematic contribution is the one arising from the first term, for which we bound

$$\frac{\lambda^{17/4}}{t^{3/2-3a}}\|\varphi_\lambda\|^3_{L^\infty_x}\lesssim t^{3a-3}\|\varphi\|^3_X\lesssim t^{-6/5}\epsilon^3 t^{C\epsilon^2}$$

The contribution arising from the second term can be immediately bounded by

$$\frac{\lambda^{13/4}}{t^{1/2-a}}\|\varphi_\lambda\|^3_{L^\infty_x}\lesssim t^{a-2}\|\varphi\|^3_X\lesssim t^{-7/5}\epsilon^3 t^{C\epsilon^2}\lesssim t^{-6/5}\epsilon^3 t^{C\epsilon^2}$$

The L^2_v-bound is similar.

This means that we have to analyze

$$\begin{aligned}q(1)\langle\partial_x(\chi_\lambda(x/t)^3(\phi'(x/t))^2e^{it\phi(x/t)}),\mathbf{u}^v_\lambda\rangle &= q(1)\langle(\chi_\lambda(x/t)\phi'(x/t))^3e^{it\phi(x/t)},\mathbf{u}^v_\lambda\rangle\\ &+q(1)t^{-1}\langle(3\chi_\lambda(x/t)^2\chi'_\lambda(x/t)(\phi'(x/t))^2+2\chi_\lambda(x/t)^3\phi'(x/t)\phi''(x/t)e^{it\phi(x/t)}),\mathbf{u}^v_\lambda\rangle,\end{aligned}$$

where the last contribution can be immediately shown to be an acceptable error by using the condition $(t,v)\in\mathscr{D}$. Further, we may replace $\mathbf{u}^v_\lambda$ by $\mathbf{u}^v$. To see this, from the proof of Lemma 5.8 in [86], we have

$$|P_{\neq\lambda}\mathbf{u}^v|\lesssim\lambda^{1/2}(1+|y|)^{-1-\delta}t^{-1-\delta}\lambda^{-1-\delta},\qquad y=(x-vt)|ta''(\xi_v)|^{-\frac{1}{2}},$$

and

$$|(\chi_\lambda(x/t)\phi'(x/t))^3e^{it\phi(x/t)}|\lesssim\lambda^3.$$

Thus,

$$\left|\left\langle(\chi_\lambda(x/t)\phi'(x/t))^3e^{it\phi(x/t)},P_{\neq\lambda}\mathbf{u}^v\right\rangle\right|\lesssim\lambda^3\lambda^{-1/2-\delta}t^{-1-\delta}t^{1/2}\lambda^{-1/2}\lesssim t^{-1/2-\delta}\lambda^{2-\delta},$$

which along with the condition $(t,v)\in\mathscr{D}$ shows that this is an acceptable error.

As $\mathbf{u}^v$ is supported in the region $\left|\frac{x}{t}-v\right|\lesssim t^{-1/2}\lambda^{-1/2}$, we can replace x/t by v in $\chi_\lambda(x/t)\phi'(x/t)$, with acceptable errors. As $\chi_\lambda(v)=1$, the remaining term is now

$$iq(1)(\chi_\lambda(v)\xi_v)^3\left\langle e^{it\phi(x/t)},\mathbf{u}^v\right\rangle=t^{\frac{1}{2}}iq(\xi_v)\xi_v,$$

as desired. $\square$

3.30.1 Closing the bootstrap argument

We recall that

$$\|\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-(1-\delta-\delta_1)}\|\varphi\|_X \lesssim \frac{1}{\sqrt{t}}\lambda^{-(1-\delta-\delta_1)}\epsilon t^{C\epsilon^2},$$

when $\lambda \le 1$ and

$$\|\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-(\frac{3}{2}+3\delta/2)}\|\varphi\|_X \lesssim \frac{1}{\sqrt{t}}\lambda^{-(\frac{3}{2}+3\delta/2)}\epsilon t^{C\epsilon^2},$$

when $\lambda > 1$.

Thus, if $t \lesssim \lambda^N$ when $\lambda > 1$, and if $t \lesssim \lambda^{-N}$ when $\lambda \le 1$, where N can be chosen appropriately, we get the desired bounds. We are left to analyze the cases $t \gtrsim \lambda^N$ when $\lambda > 1$, and $t \gtrsim \lambda^{-N}$ when $\lambda \le 1$.

We recall that in the elliptic region,

$$\||D_x|^\delta((1-\chi_\lambda)\varphi_\lambda(x))\|_{L^\infty_x} \lesssim \frac{\lambda^{\delta-1/2}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{1/2-\delta}+\lambda^{-\delta}}{t}\epsilon t^{C^2\epsilon^2}$$
$$\||D_x|^{\frac{1}{2}+\delta}\partial_x((1-\chi_\lambda)\varphi_\lambda(x))\|_{L^\infty_x} \lesssim \frac{\lambda^{1+\delta}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{1+\delta}+\lambda^{1/2+\delta}}{t}\epsilon t^{C^2\epsilon^2}$$

which give the desired bounds in both cases. It remains to bound the non-elliptic region $\chi_\lambda\varphi_\lambda$. We recall that, if $x/t \in J_\lambda$, and $r(t,x) = \chi_\lambda\varphi_\lambda(t,x) - \frac{1}{\sqrt{t}}\chi_\lambda\gamma(t,x/t)e^{it\phi(x/t)}$,

$$t^{1/2}\|r^\lambda\|_{L^\infty_x} \lesssim t^{-1/4}\lambda^{-5/4}(1+\lambda^{-1})\epsilon t^{C\epsilon^2}$$

When $\lambda \le 1$ and $t \gtrsim \lambda^{-N}$, we note that

$$t^{-1/4}\lambda^{-5/4}(1+\lambda^{-1})\epsilon t^{C\epsilon^2} \lesssim \lambda^{-(1-\delta-\delta_1)}\epsilon.$$

This is true because it is equivalent to

$$\lambda^{-(1/4+\delta+\delta_1)}(1+\lambda^{-1}) \lesssim t^{1/4-C\epsilon^2}.$$

When $\lambda > 1$ and $t \gtrsim \lambda^N$, we can see that

$$t^{-1/4}\lambda^{-5/4}(1+\lambda^{-1})\epsilon t^{C\epsilon^2} \lesssim \lambda^{-(\frac{3}{2}+3\delta/2)}\epsilon.$$

This is true because it is equivalent to

$$\lambda^{1/4+3\delta/2} \lesssim t^{1/4-C\epsilon^2}.$$

This means that we only need the bounds

$$|\gamma(t,v)| \lesssim \epsilon\lambda^{-(1-\delta-\delta_1)}$$

when $\lambda \le 1$, and

$$|\gamma(t,v)| \lesssim \epsilon\lambda^{-(\frac{3}{2}+3\delta/2)}$$

when $\lambda > 1$. By initializing at time $t = 1$, up to which the bounds are known to be true from the energy estimates, and by using Proposition 3.30, we reach the desired conclusion.

3.31 Modified scattering

In this section we discuss the modified scattering behaviour of the global solutions constructed in Section 3.21. We recall that the solutions of (3.1.3) have conserved mass (as llng as it is well-defined):

Proposition 3.32 (Proposition 7.1, [6])**.** For solutions φ of (3.1.3), $\|\varphi(t)\|_{L^2}^2$ is conserved in time.

Recall the asymptotic equation

$$\dot{\gamma}(t,v) = iq(\xi_v)\xi_v t^{-1} \left|\gamma(t,v)\right|^2 \gamma(t,v) + f(t,v),$$

As $t \to \infty$, $\gamma(t,v)$ converges to the solution of the equation

$$\dot{\tilde{\gamma}}(t,v) = iq(\xi_v)\xi_v t^{-1}\tilde{\gamma}(t,v)|\tilde{\gamma}(t,v)|^2,$$

whose solution is

$$\tilde{\gamma}(t,v) = W(v)e^{iq(\xi_v)\xi_v \ln(t)|W(v)|^2}$$

We can immediately see that $W(v)$ is well-defined, as $|W(v)| = |\tilde{\gamma}(t,v)|$, which is a constant, and

$$W(v) = \lim_{s\to\infty} \tilde{\gamma}(e^{2s\pi/(q(\xi_v)\xi_v|W(v)|^2)}, v).$$

Corollary 3.33. Let $v \in J_\lambda$. Under the assumption $(t,v) \in \mathscr{D}$, we have the asymptotic expansions

$$\|\gamma(t,v) - W(v)e^{iq(\xi_v)\xi_v \log t|W(v)|^2}\|_{L^\infty(J_\lambda)} \lesssim \lambda^{-\delta} g(\lambda) t^{-\delta+C^2\epsilon^2}\epsilon. \tag{3.33.1}$$

$$\|\gamma(t,v) - W(v)e^{iq(\xi_v)\xi_v \log t|W(v)|^2}\|_{L^2(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-3/2}) t^{-\delta+C^2\epsilon^2}\epsilon. \tag{3.33.2}$$

Proof. This is an immediate consequence of Proposition 3.30. □

Proposition 3.34. Under the assumption

$$\|\varphi_0\|_X \lesssim \epsilon \ll 1,$$

the asymptotic profile W defined above satisfies

$$\|e^{-\frac{v(1+\delta)}{2}} e^{|v|(1/2+\delta/4)} |D_v|^{1-C_1\epsilon^2} W(v)\|_{L^2_v} \lesssim \epsilon.$$

Moreover, when $s_0 = 0$, we also have $\|W(v)\|_{L^2_v} \lesssim \epsilon$.

Proof. We fix λ, and let $t \gtrsim \max\{1, \lambda^{-1}\}$. From Corollary 3.33 we know that

$$\|W(v) - e^{-iq(\xi_v)\xi_v \log t|\gamma(t,v)|^2}\gamma(t,v)\|_{L^2_v(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-3/2})t^{-\delta+C^2\epsilon^2}\epsilon.$$

From the product and chain rules with Lemma 3.24, we have

$$\left\|\partial_v \left(e^{-iq(\xi_v)\xi_v \log t|\gamma(t,v)|^2}\gamma(t,v)\right)\right\|_{L^2_v(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-2})\log(t)\epsilon t^{C^2\epsilon^2}.$$

Putting these together, we find that for $t \gtrsim \max\{1, \lambda^{-1}\}$,

$$W(v) = O_{\dot{H}^1_v(J_\lambda)}(\lambda^{-\delta}(1+\lambda^{-2})\log(t)\epsilon t^{C^2\epsilon^2}) + O_{L^2_v(J_\lambda)}(\lambda^{-\delta}(1+\lambda^{-3/2})t^{-\delta+C^2\epsilon^2}\epsilon).$$

By interpolation, this will imply that for C_1 large enough we have

$$\|W(v)\|_{\dot{H}^{1-C_1\epsilon^2}_v(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-2})\epsilon.$$

By dyadic summation over $\lambda \geq 1$ and $\lambda \leq 1$,

$$\|e^{-\frac{v(1+\delta)}{2}}e^{|v|(1/2+\delta/4)}|D_v|^{1-C_1\epsilon^2}W(v)\|_{L^2_v} \lesssim \epsilon.$$

The last part immediately follows from the conservation of mass. $\square$

Chapter 4

The generalized surface quasi-geostrophic (gSQG) front equations

4.1 Introduction

This chapter is going to be concerned with the contents of preprint [4], and is companion to Chapter 3 in the same way as [4] is companion to [3]. The generalized surface quasi-geostrophic (gSQG) equations are a one parameter family of active scalar equations parameterized by a transport term, given by

$$\theta_t + u \cdot \nabla\theta = 0, \qquad u = (-\Delta)^{-1+\frac{\alpha}{2}}\nabla^\perp\theta, \qquad \alpha \in [0, 2). \tag{4.1.1}$$

Here, θ represents a scalar evolution on $\mathbb{R}^2$, $(-\Delta)^{-1+\frac{\alpha}{2}}$ is a fractional Laplacian, and $\nabla^\perp$ is given by $\nabla^\perp = (-\partial_y, \partial_x)$.

The case $\alpha = 0$ corresponds to the two-dimensional incompressible Euler equation, while the case $\alpha = 1$ gives the surface quasi-geostrophic equation (SQG) equation. The latter arises as a model for quasi-geostrophic flows confined to a surface in atmospheric and oceanic science. It also shares some similarities with the three dimensional incompressible Euler equation, and thus is often used as a simplified model problem. In particular, the question of finite time singularity formation remains open for both equations. For further analysis of the SQG equation, see Resnick [120].

Front solutions to (4.1.1) are solutions of the form

$$\theta(t, x, y) = \begin{cases} \theta_+ & \text{if } y > \varphi(t, x), \\ \theta_- & \text{if } y < \varphi(t, x), \end{cases}$$

where the front refers to the graph $y = \varphi(t, x)$ with $x \in \mathbb{R}$. Front solutions are closely related to patch solutions, which have the form

$$\theta(t,x,y) = \begin{cases} \theta_+ & \text{if } (x,y) \in \Omega(t), \\ \theta_- & \text{if } (x,y) \notin \Omega(t), \end{cases}$$

where $\Omega(t)$ is a bounded, simply connected domain.

When $\alpha \in (1,2)$, the derivation and analysis of contour dynamics equations governing fronts and patches do not differ substantially. However, when $\alpha \in [0,1]$, the derivation of the equations for fronts has additional complexities relative to the case of patches, arising from the slow decay of Green's functions. The derivation in this range was provided by Hunter-Shu [74] via a regularization procedure, and again by Hunter-Shu-Zhang in [76]. In the generalized $\alpha \neq 1$ case, the equation takes the form

$$\begin{aligned} (\partial_t - c(\alpha)|D_x|^{\alpha-1}\partial_x)\varphi &= Q(\varphi, \partial_x\varphi), \\ \varphi(0,x) &= \varphi_0(x), \end{aligned} \tag{4.1.2}$$

while in the SQG case $\alpha = 1$, the equation takes the form

$$\begin{aligned} (\partial_t - 2\log|D_x|\partial_x)\varphi &= Q(\varphi, \partial_x\varphi), \\ \varphi(0,x) &= \varphi_0(x). \end{aligned} \tag{4.1.3}$$

Here, φ is a real-valued function $\varphi : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $c(\alpha)$ denotes the constant

$$\begin{aligned} c(\alpha) &= -2\sin\left(\frac{\pi(2-\alpha)}{2}\right)\Gamma(1-\alpha), & \alpha &\in (0,2), \\ c(\alpha) &= -\frac{1}{2}, & \alpha &= 0, \end{aligned} \tag{4.1.4}$$

and the nonlinearity Q is given in the two cases $\alpha \in (0,2)$ and $\alpha = 0$ respectively by

$$Q(f,g)(x) = \int \left(\frac{1}{|y|^\alpha} - \frac{1}{(y^2 + (f(x+y) - f(x))^2)^{\frac{\alpha}{2}}}\right) \cdot (g(x+y) - g(x))\, dy \tag{4.1.5}$$

and

$$Q(f,g)(x) = \int \frac{1}{2\pi} \log\left(1 + \left(\frac{f(x+y) - f(x)}{y}\right)^2\right) \cdot (g(x+y) - g(x))\, dy. \tag{4.1.6}$$

The nonlinearity Q can be written more succinctly using difference quotients,

$$Q(f,g)(x) = \int \frac{1}{|y|^{\alpha-1}} F(\delta^y f) \cdot |\delta|^y g \, dy, \tag{4.1.7}$$

where

$$\delta^y f(x) = \frac{f(x+y) - f(x)}{y}, \qquad |\delta|^y g(x) = \frac{g(x+y) - g(x)}{|y|},$$

and

$$F(s) = 1 - \frac{1}{(1+s^2)^{\frac{\alpha}{2}}} \quad \text{when } \alpha \in (0,2), \qquad F(s) = \frac{1}{2\pi}\log(1+s^2) \quad \text{when } \alpha = 0.$$

The equations (4.1.2) and (4.1.3) are invariant under the scaling

$$t \to \kappa^{\alpha} t, \qquad x \to \kappa x, \qquad \varphi \to \kappa \varphi,$$

which implies that $\dot{H}^{\frac{3}{2}}(\mathbb{R})$ is the corresponding critical Sobolev space.

In the case of SQG patches, the first local well-posedness results were obtained by Rodrigo [121] for initial data in C^{∞}. Gancedo-Nguyen-Patel later proved in [55] that under a suitable parametrization, the contour dynamics evolution is locally well-posed in $H^s(\mathbb{T})$ when $s > 2$. For the generalized SQG family, the first local well-posedness results were obtained by Gancedo [54] in the case $\alpha \in (0,1]$, for which they showed that the problem is locally well-posed in $H^3(\mathbb{T})$. Later on, Gancedo-Patel analyzed the gSQG case with $\alpha \in (0,2)$ and $\alpha \neq 1$ in [56], where they in particular obtained local well-posedness in H^2 for $\alpha \in (0,1)$ and H^3 for $\alpha \in (1,2)$. For a more recent result on enhanced lifespan for α-patches, see Berti-Cuccagna-Gancedo-Scrobogna [20].

Patches in the 2D Euler case $\alpha = 0$ correspond to a special kind of Yudovich [98] solution, also referred to as Euler vortex patches. Bertozzi [22] showed that they are locally well-posed in the space $C^{1,\delta}$. By using paradifferential calculus, Chemin [27] proved that given a patch solution whose boundary is $C^{k,\mu}$ at the initial time, its regularity persists for all times. Bertozzi-Constantin [21] soon obtained another proof based on a level set approach, and yet another proof was subsequently obtained by Serfati [122]. Recently, Radu [119] proved global existence for Euler patch solutions. For results in Sobolev spaces, see Coutand-Shkoller [42].

For the question of ill-posedness in the context of patches, Kiselev-Luo [104] obtained some results in Sobolev spaces with exponents $p \neq 2$, as well as in Hölder spaces. In addition, Zlato**vs** [148] showed that, provided local well-posedness is known for some $\alpha \in (0, \frac{1}{2}]$, suitable initial data give rise to blow up for both bounded and unbounded patch solutions.

In the current chapter, we are interested in the well-posedness of gSQG fronts, following our previous work on the SQG case [3]. The first results in the gSQG front setting were obtained by Córdoba-Gómez-Serrano-Ionescu [39] for $\alpha \in (1,2)$, showing global well-posed for small and localized initial data in H^s, where $s > 20\alpha$.

For the cases $\alpha \in (0,1]$ spanning 2D Euler to the classical SQG, Hunter-Shu-Zhang first studied the local well-posedness for a cubic approximation of the SQG equation in [75], before establishing in [77] local well-posedness for the full SQG equation (3.1.3) with initial data in H^s with $s \geq 5$, along with global well-posedness for small, localized, and essentially smooth ($s \geq 1200$) initial data. These results were later extended to the gSQG cases $\alpha \in (0,1)$, while also lowering the regularity threshold to $s > \frac{7}{2} + \frac{3\alpha}{2}$ [78].

Although it is typical to study global well-posedness in the context of small and localized data, we remark that the local well-posedness results of [75, 78] were also established in the context of a small data assumption, as well as a convergence condition on an expansion of the nonlinearity $Q(\varphi, \partial_x \varphi)$ from (3.1.3). There, the purpose of the small data was to ensure that the modified energies used in the proof were coercive.

In the SQG case $\alpha = 1$, the authors lowered both the local and global well-posedness regularity thresholds, to $s > \frac{5}{2}$ and $s > 4$ respectively [6], while removing the small data and convergence conditions from the local well-posedness result. Subsequently, by observing a null structure satisfied by the SQG equation (4.1.3), the authors further improved the low regularity thresholds to $s > 2$ and $s > 3$ respectively [3], while also improving the low frequency threshold to $\dot{H}^{s_0}$ for any $s_0 < \frac{3}{2}$. In particular, this result establishes local well-posedness without requiring control at the level of L^2.

Our current objective is to consider the generalized SQG family $\alpha \in [0, 2)$, to prove lower regularity local and global well-posedness results which parallel those of the SQG case $\alpha = 1$. Our contributions include

- obtaining a local well-posedness result in a significantly lower regularity setting, at $\frac{\alpha}{2}$ derivatives above scaling, by making use of a null structure exhibited by the generalized family of equations, and
- proving low regularity global well-posedness by using the wave packet testing method of Ifrim-Tataru (see for instance [84, 86]).

4.1.1 Main results

Similar to the SQG setting considered in [3], a key observation of the current article is that the gSQG equation (4.1.2) exhibits a resonance structure when $\alpha \in (0, 1) \cup (1, 2)$. More precisely, we can approximate the nonlinearity Q by

$$Q(\varphi, v) \approx \Omega(\psi, v), \qquad \psi := \partial_x^{-1} F(\varphi_x) \tag{4.1.8}$$

where Ω is a bilinear form whose symbol is given by the resonance function

$$\Omega(\xi_1, \xi_2) = \frac{1}{\alpha}(\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2)), \qquad \omega(\xi) = c(\alpha) i\xi|\xi|^{\alpha-1}.$$

This is meaningful because ψ solves an equation similar to φ (see Proposition 4.14, part b)).

This structure enables us to use normal form methods to obtain $b$alanced energy estimates, which use control norms with an even balance of derivatives,

$$\frac{d}{dt} E^{(s)}(\varphi) \lesssim_A B^2 \cdot E^{(s)}(\varphi),$$

where

$$A = \|\partial_x \varphi\|_{L^\infty}, \qquad B = \|\partial_x \varphi\|_{B^{\frac{\alpha}{2}}_{\infty,2} \cap BMO^{\frac{\alpha}{2}}}. \tag{4.1.9}$$

We can now state our main local well-posedness result:

Theorem 4.1.1. Let $\alpha \in [0,2)$. Equation (4.1.2) is locally well-posed for initial data in $\dot{H}^{s_0} \cap \dot{H}^s$ with $s_0 < \frac{3}{2}$ and

$$
\begin{aligned}
s &> \frac{\alpha+3}{2} \qquad \text{if} \quad \alpha = 0, 1,\\
s &\geq \frac{\alpha+3}{2} \qquad \text{if} \quad \alpha \in (0,1) \cup (1,2).
\end{aligned}
$$

Precisely, for every $R > 0$, there exists $T = T(R) > 0$ such that for any $\varphi_0 \in \dot{H}^{s_0} \cap \dot{H}^s$ with $\|\varphi_0\|_{\dot{H}^{s_0} \cap \dot{H}^s} < R$, the Cauchy problem (3.1.3) has a unique solution $\varphi \in C([0,T], \dot{H}^{s_0} \cap \dot{H}^s)$. Moreover, the solution map $\varphi_0 \mapsto \varphi$ from $\dot{H}^{s_0} \cap \dot{H}^s$ to $C([0,T], \dot{H}^{s_0} \cap \dot{H}^s)$ is continuous.

Remark 4.1.1. The SQG case $\alpha = 1$ is the subject of Chapter 3 and of the paper [3], while the 2D Euler case $\alpha = 0$ is addressed in Appendix 4.41.

For the main subject of the current chapter, the cases $\alpha \in (0,1) \cup (1,2)$, the control norms (4.1.9) allow us to obtain the local well-posedness in the endpoint case $s = \frac{\alpha+3}{2}$. In contrast, in the SQG and 2D Euler cases, we only prove the result in the case $s > 2$ and $s > 3/2$ respectively, due to the logarithmic loss generated by the dispersion relation and nonlinearities. Precisely, for comparison, we recall that in the analysis of the SQG case in [3], we used the control parameter

$$
B = \|\partial_x \varphi\|_{C^{\frac{1}{2}+}}.
$$

Remark 4.1.2. For the sake of simplicity, we assume that the parameter A is small. This assumption can be removed with a more careful definition of the paraproduct, at the expense of having to deal with more technical details; see for instance [6].

Remark 4.1.3. We also note that up to the endpoint, this result is sharp as long as energy estimates are concerned. In order to see this, we consider the nonlinearity $\partial_x A_\varphi v$, which appears in the linearized equation, with linearized variables v. We assume that we want to bound the map $v \to \partial_x A_\varphi v$ in L^2, and that for the time being, we can redistribute the arising derivatives in $\partial_x A_\varphi v$ as we wish. We have

$$
\partial_x A_\varphi v = \partial_x \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi) |\delta|^y v \, dy,
$$

and we can see that, morally speaking, we have $\alpha + 2$ derivatives, that we want to distribute onto the two φ factors (we recall that F is quadratic). In order to achieve this by requiring as little regularity from φ as possible, it is clear that the $\alpha+2$ derivatives have to be distributed equally between the two factors. This means that we would need the norm $\|"|D_x|^{1+\frac{\alpha}{2}}"\varphi\|_{L^\infty_x}$ to be finite, which imposes the condition $s \geq \dfrac{\alpha+3}{2}$.

Normal forms were introduced by Shatah [123], who used them to prove results on the long-time dynamics of solutions to dispersive equations. Unfortunately, this method cannot be applied directly to quasilinear problems, because the resulting transformations would be unbounded. To address this issue, several approaches have been introduced; we rely on two in this chapter. The first consists of carrying out the analysis in a paradifferential manner, and was introduced by Alazard-Delort [12] in a paradiagonalization argument to obtain Sobolev estimates for the solutions of the water waves equations in the Zakharov formulation. This approach was also subsequently employed by Ifrim-Tataru [85] to obtain a new proof of L^2 global well-posedness for the Benjamin-Ono equation, a result first obtained in [89].

The second method consists of employing modified energies instead of applying the direct normal form at the level of the equation. This procedure was introduced by Hunter-Ifrim-Tataru-Wong [71] to study the long time behavior of solutions to the Burgers-Hilbert equation.

These approaches were first combined in order to study the low regularity well-posedness for quasilinear models by the first author together with Ifrim-Tataru [7] for the gravity water waves system, through the proof of *balanced energy estimates*. This type of estimate was then later used together with Strichartz estimates to obtain low regularity well-posedness results for the time-like minimal surface problem in the Minkowski space [8].

We next consider global well-posedness for small and localized data. To describe localized solutions, we define the operator

$$L = x + t\alpha c(\alpha)|D_x|^{\alpha-1},$$

which commutes with the linear flow $\partial_t - c(\alpha)|D_x|^{\alpha-1}\partial_x$, and at time $t = 0$ is simply multiplication by x. Then we define the time-dependent weighted energy space

$$\|\varphi\|_X := \|\varphi\|_{\dot{H}^{s_0}\cap\dot{H}^s} + \|L\partial_x\varphi\|_{L^2},$$

where $s > \alpha + 2$ and $s_0 < \min\{1, \alpha\}$. To track the dispersive decay of solutions, we define the pointwise control norm

$$\|\varphi\|_Y := \||D_x|^{1-\delta}\langle D_x\rangle^{\frac{\alpha}{2}+2\delta}\varphi\|_{L^\infty_x}.$$

Theorem 4.1.2. Consider data φ_0 with

$$\|\varphi_0\|_X \lesssim \epsilon \ll 1.$$

Then the solution φ to (4.1.2) for $\alpha > 0$ with initial data φ_0 exists globally in time, with energy bounds

$$\|\varphi(t)\|_X \lesssim \epsilon t^{C\epsilon^2}$$

and pointwise bounds

$$\|\varphi(t)\|_Y \lesssim \epsilon\langle t\rangle^{-\frac{1}{2}}.$$

Further, the solution φ exhibits a modified scattering behavior, with an asymptotic profile $W \in H^{1-C_1\epsilon^2}(\mathbb{R})$, in a sense that will be made precise in Section 4.37.

Regarding the question of large data global well-posedness, the absence of splash singularities in the case of patches was first proved by Gancedo-Strain [57]. Recently, Kiselev-Luo [105] sharpened the criterion ruling out splash singularities. More precisely, they improved the double exponential bound of Gancedo-Strain [57] to just exponential in time.

The question of extending our results to the non-graph case is interesting and open. Our current approach does not provide an answer, as a different parametrization would be needed.

4.1.2 Outline of the chapter.

The organization of the chapter follows closely that of the SQG case covered in Chapter 2 and in the paper [3]. Here, the main novelties include improvements which allow us to obtain the endpoint regularity $s = \frac{\alpha+3}{2}$, as well as a generalized modified energy and normal form transformation.

In Section 4.2, we introduce notations and establish preliminary estimates for paradifferential calculus and difference quotients.

In Section 4.9, we describe the null structure of equation (4.1.2) and of its linearization,

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v = \partial_x Q(\varphi, v). \tag{4.1.10}$$

We also define the paradifferential flow associated to (4.1.10), which will play a key role in the analysis.

In Section 4.15 we reduce the energy estimates and well-posedness for the linearized equation (4.1.10) to the case of the inhomogeneous paradifferential flow. The main challenge is ensuring that the paradifferential error terms satisfy cubic balanced energy estimates, which we achieve by performing a normal form analysis.

In Section 4.18 we prove energy estimates for the paradifferential flow. To achieve this, we construct a paradifferential modified energy functional. Here, we note that unlike in the SQG case $\alpha = 1$ [3], the normal form underlying the energy functional coincides with the normal form correction of the previous section only to first order, the latter being just a linearization of the former.

In Section 4.21, we obtain higher order energy estimates. As the resulting commutators are quadratic and thus not perturbative, they pose an additional obstacle. To address this, we perform a two-step change of variables. First, we eliminate the highest order terms by using a Jacobian exponential conjugation, at the cost of creating a number of lower order factors. Then, we carry out a normal form analysis to eliminate these remaining non-perturbative terms.

In Section 4.24, we provide the proof of our local well-posedness result. Here we construct rough solutions as the unique limits of smooth solutions, controlled by frequency envelopes. This technique was introduced by Tao in [135], who used it to obtain more accurate information on the evolution of the energy distribution between dyadic frequencies under nonlinear flows. This method is systematically presented in the context of local well-posedness theory for quasilinear problems by Ifrim-Tataru in their expository paper [87].

In Section 4.25 we prove the global well-posedness part of Theorem 4.1.2 and the dispersive bounds on the resulting solutions, by using the wave packet testing method of Ifrim-Tataru [86]. In Section 4.37, we prove the modified scattering behavior, completing the proof of Theorem 4.1.2.

Finally, in the appendix we prove the local well-posedness result in the case $\alpha = 0$, which corresponds to Euler fronts. Here, we note that due to the degenerate character of the dispersion relation, the methods that we used cannot be applied to obtain an analogue of the global well-posedness result.

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4.2 Notations and classical estimates

In this section we introduce some notations and classical estimates that we use throughout the article. These include paradifferential calculus and difference quotient estimates.

4.2.1 Paradifferential operators and paraproducts

Let χ be an even smooth function such that $\chi = 1$ on $[-\frac{1}{20}, \frac{1}{20}]$ and $\chi = 0$ outside $[-\frac{1}{10}, \frac{1}{10}]$, and define

$$\tilde{\chi}(\theta_1, \theta_2) = \chi\left(\frac{|\theta_1|^2}{M^2 + |\theta_2|^2}\right).$$

Given a symbol $a(x,\eta)$, we use the above cutoff $\tilde{\chi}$ to define an M-dependent paradifferential quantization of a by (see also [11])

$$\widehat{T_a u}(\xi) = (2\pi)^{-1} \int \hat{P}_{>M}(\xi)\tilde{\chi}\left(\xi - \eta, \xi + \eta\right) \hat{a}(\xi - \eta, \eta) \hat{P}_{>M}(\eta)\hat{u}(\eta)\, d\eta,$$

where the Fourier transform of the symbol $a(x,\eta)$ is taken with respect to the first argument.

This quantization was employed in [6], where the parameter M was introduced to ensure the coercivity of the modified quasilinear energy without relying on a small data assumption. We recall in particular that in the case of a paraproduct, where $a = a(x)$, T_a is self-adjoint.

We recall the following classical estimates of paradifferential calculus, and defer to the corresponding sections and statements in [7, 6] for the proofs.

The following two commutator-type estimates are exact reproductions of statements from Lemmas 2.4 and 2.6 in Section 2 of [7], respectively:

Lemma 4.3 (Para-commutators)**.** Assume that $\gamma_1, \gamma_2 < 1$. Then we have

$$\|T_f T_g - T_g T_f\|_{\dot{H}^s \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{BMO} \||D|^{\gamma_2} g\|_{BMO}, \tag{4.3.1}$$

$$\|T_f T_g - T_g T_f\|_{\dot{B}^s_{\infty,\infty} \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{L^2} \||D|^{\gamma_2} g\|_{BMO}. \tag{4.3.2}$$

A bound similar to the one in Lemma (4.3.1) holds in the Besov scale of spaces, namely from $\dot{B}^s_{p,q}$ to $\dot{B}^{s+\gamma_1+\gamma_2}_{p,q}$ for real s and $1 \leq p, q \leq \infty$.

The next paraproduct estimate, see [7, Lemma 2.5], directly relates multiplication and paramultiplication:

Lemma 4.4 (Para-products)**.** Assume that $\gamma_1, \gamma_2 < 1$. Then

$$\|T_f T_g - T_{T_f g}\|_{\dot{H}^s \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{BMO} \||D|^{\gamma_2} g\|_{BMO}. \tag{4.4.1}$$

If in addition $\gamma_1 + \gamma_2 \geq 0$,

$$\|T_f T_g - T_{fg}\|_{\dot{H}^s \to \dot{H}^{s+\gamma_1+\gamma_2}} \lesssim \||D|^{\gamma_1} f\|_{BMO} \||D|^{\gamma_2} g\|_{BMO}. \tag{4.4.2}$$

Similar bounds hold in the Besov scale of spaces, namely from $\dot{B}^s_{p,q}$ to $\dot{B}^{s+\gamma_1+\gamma_2}_{p,q}$ for real s and $1 \leq p, q \leq \infty$.

Next, we recall the following Moser-type estimate; see for instance [6].

Theorem 4.4.1 (Moser)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function with $F(0) = 0$, $\gamma_1 + \gamma_2 = s > 0$, and

$$R(v) = F(v) - T_{F'(v)} v.$$

Then

$$\|R(v)\|_{\dot{W}^{s,\infty}} \lesssim_{\|v\|_{L^\infty}} \||D|^{\gamma_1} v\|_{BMO \cap \dot{B}^0_{\infty,2}} \||D|^{\gamma_2} v\|_{BMO \cap \dot{B}^0_{\infty,2}}. \tag{4.4.3}$$

From this Moser estimate, we obtain the following Moser-type paraproduct estimate:

Lemma 4.5. Assume that $0 \leq \gamma_1 + \gamma_2 < 1$. Let F be a smooth function. Then we have

$$\|T_{\partial_x F(f)} - T_{F'(f)} T_{f_x}\|_{\dot{H}^s \to \dot{H}^{s+\gamma_1+\gamma_2-1}} \lesssim \||D|^{\gamma_1} f\|_{BMO} \||D|^{\gamma_2} f\|_{BMO}. \tag{4.5.1}$$

Remark 4.5.1. We need this estimate only in the $\alpha < 1$ case, to shift derivatives between both the low frequency paraproducts as well as the high frequency argument. In the $\alpha > 1$ case, it always suffices to balance derivatives between just the low frequency paraproducts using Lemma 4.4 directly.

Proof. We collect the errors as we transition from $T_{\partial_x F(f)}$ to $T_{F'(f)}T_{f_x}$, each of which can be bounded by the right hand side of (4.5.1). We apply the Moser estimate of Theorem 4.4.1 to replace $\partial_x F(f)$ with $\partial_x T_{F'(f)} f$. To address this latter term, we expand via the product rule,

$$\partial_x T_{F'(f)} f = T_{\partial_x F'(f)} f + T_{F'(f)} \partial_x f.$$

A second application of the Moser estimate then estimates the contribution from $T_{\partial_x F'(f)} f$, which altogether leaves us with

$$T_{T_{F'(f)} f_x} - T_{F'(f)} T_{f_x}.$$

By the first estimate of Lemma 4.4, it follows that this error is acceptable. $\square$

4.5.1 Difference quotients

Recall that we denote difference quotients by

$$\delta^y h(x) = \frac{h(x+y) - h(x)}{y}, \qquad |\delta|^y h(x) = \frac{h(x+y) - h(x)}{|y|},$$

and define the smooth function

$$F(s) = 1 - \frac{1}{(1+s^2)^{\frac{\alpha}{2}}},$$

which in particular vanishes to second order at $s = 0$, satisfying $F(0) = F'(0) = 0$. Using this notation, we may express

$$Q(\varphi, v)(t,x) = \int |y|^{1-\alpha} F(\delta^y \varphi(t,x)) \cdot |\delta|^y v(t,x)\, dy.$$

In addition, to facilitate the normal form analysis in later sections, we denote

$$\psi := \partial_x^{-1} F(\partial_x \varphi), \qquad J = (1 - F(\partial_x \varphi))^{-1} = (1 - \partial_x \psi)^{-1},$$

where we fix the antiderivative such that $\psi(-\infty) = 0$.

We have the following estimate which allows the balancing of up to $2-\alpha$ derivatives over multilinear averages of difference quotients:

Lemma 4.6. Let $i = \overline{1,n}$ and $p_i, r \in [1,\infty]$ and $\alpha_i, \beta_i \in [0,1]$ satisfying

$$\sum_i \frac{1}{p_i} = \frac{1}{r}, \qquad n+\alpha-2 < \sum_i \alpha_i \le n, \qquad 0 \le \sum_i \beta_i < n+\alpha-2.$$

Then

$$\left\| \int \frac{1}{|y|^{\alpha-1}} \prod_i \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \prod_i \| |D|^{\alpha_i} f_i \|_{L^{p_i}} + \prod_i \| |D|^{\beta_i} f_i \|_{L^{p_i}}.$$

Proof. We write

$$\int \frac{1}{|y|^{\alpha-1}} \prod_i \delta^y f_i \, dy = \int_{|y|\le 1} + \int_{|y|>1}.$$

For the former integral, we have by Hölder

$$\left\| \int_{|y|\le 1} \frac{1}{|y|^{\alpha-1}} \prod_i \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \int_{|y|\le 1} \frac{1}{|y|^{n+\alpha-1-\sum \alpha_i}} \prod_i \| |D|^{\alpha_i} f_i \|_{L^{p_i}} \, dy \lesssim \prod_i \| |D|^{\alpha_i} f_i \|_{L^{p_i}}.$$

The latter integral is treated similarly,

$$\left\| \int_{|y|>1} \frac{1}{|y|^{\alpha-1}} \prod_i \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \int_{|y|>1} \frac{1}{|y|^{n+\alpha-1-\sum \beta_i}} \prod_i \| |D|^{\beta_i} f_i \|_{L^{p_i}} \, dy \lesssim \prod_i \| |D|^{\beta_i} f_i \|_{L^{p_i}}.$$

∎

The above lemma may be sharpened using Besov spaces. First, we recall the following difference quotient representation of the Besov space $\dot{B}^s_{p,r}$ [17, Theorem 2.36]:

Lemma 4.7. Let $s \in (0,1)$ and $(p,r) \in [1,\infty]^2$. Then

$$\|u\|_{\dot{B}^s_{p,r}} \approx \left\| \frac{\|u(x+y)-u(x)\|_{L^p_x}}{|y|^s} \right\|_{L^r_y(\mathbb{R},\frac{1}{|y|})}.$$

We apply this to establish a Besov version of the estimate on multilinear averages:

Lemma 4.8. Let $i = \overline{1,n}$ and $p_i, r \in [1,\infty]$ and $\alpha_i \in [0,1)$ satisfying

$$\sum_i \frac{1}{p_i} = \frac{1}{r}, \qquad \sum_i \frac{1}{q_i} = 1, \qquad \sum_i \alpha_i = 2-\alpha.$$

Then

$$\left\| \int \frac{1}{|y|^{\alpha-1}} \prod_i \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \prod_i \|f_i\|_{\dot{B}^{1-\alpha_i}_{p_i,q_i}}.$$

Proof. By the triangle and Hölder's inequalities, we have

$$\left\| \int \frac{1}{|y|^{\alpha-1}} \prod_i \delta^y f_i \, dy \right\|_{L^r_x} \lesssim \int \frac{1}{|y|} \left\| \prod_i |y|^{\alpha_i} \delta^y f_i \right\|_{L^r_x} dy \lesssim \prod_i \int \frac{1}{|y|} \||y|^{\alpha_i} \delta^y f_i\|_{L^{p_i}_x} \, dy$$
$$\lesssim \prod_i \| \||y|^{\alpha_i} \delta^y f_i\|_{L^{p_i}_x} \|_{L^{q_i}_y(\mathbb{R}, \frac{1}{|y|})} \lesssim \prod_i \|f_i\|_{\dot{B}^{1-\alpha_i}_{p_i,q_i}}.$$

□

4.9 The null structure and paradifferential equation

In this section and the next, we reduce energy estimates for the linearized equation (4.1.10),

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v = \partial_x Q(\varphi, v),$$

to energy estimates for a corresponding paradifferential equation.

This can be achieved by treating (4.1.10) as a paradifferential equation with a perturbative source, where the main task is to paralinearize the cubic term $\partial_x Q(\varphi, v)$. Moreover, we are interested in carrying out this process in a manner such that the perturbative errors satisfy $b$alanced estimates. Precisely, we obtain estimates that only involve the control norms (4.1.9),

$$A := \|\partial_x \varphi\|_{L^\infty}, \qquad B := \|\partial_x \varphi\|_{B^{\frac{\alpha}{2}}_{\infty,2} \cap BMO^{\frac{\alpha}{2}}},$$

where A corresponds to the scaling-critical threshold, while B lies $\alpha/2$ derivatives above scaling.

Unfortunately, directly estimating the paralinearization errors does not allow us to prove estimates that are controlled only by A and B. Instead, we will use a paradifferential normal form transformation to eliminate the source terms that do not directly satisfy the desired balanced cubic estimates. In this section, we first consider various formulations of the paradifferential equation which will be useful in the following sections.

4.9.1 Null structure

Even though the principal term in the expansion of $F(\delta^y \varphi)$ is quadratic in φ (and thus $Q(\varphi, v)$ is principally cubic), estimates on derivatives of $F(\delta^y \varphi)$ do not fully capture its quadratic structure. This happens because they are limited by the cases of low-high interaction where derivatives fall solely on the high frequency variable. Consequently, in the context of proving balanced estimates, $F(\delta^y \varphi)$ behaves essentially like a linear coefficient.

On the other hand, we note that Q exhibits a null structure in the following sense. By writing

$$\Omega(\varphi, v) = \int \frac{1}{|y|^{\alpha-1}} \delta^y \varphi \cdot |\delta|^y v \, dy$$

and using the heuristic approximation

$$F(\delta^y \varphi) \approx T_{F'(\varphi_x)} \delta^y \varphi,$$

we may express Q as a quadratic form with a low frequency coefficient,

$$Q(\varphi, v) \approx T_{F'(\varphi_x)} \Omega(\varphi, v). \tag{4.9.1}$$

We then observe that the bilinear form $\Omega(\varphi, v)$ exhibits a null structure, since its symbol (abusing notation)

$$\Omega(\xi_1, \xi_2) = \int \frac{\operatorname{sgn} y}{|y|^{\alpha-1}} \cdot \frac{(e^{i\xi_1 y} - 1)(e^{i\xi_2 y} - 1)}{y^2}\, dy$$

satisfies the following resonance identity:

Lemma 4.10. We have

$$\alpha\Omega(\xi_1, \xi_2) = \omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2), \qquad \omega(\xi) = c(\alpha) i\xi |\xi|^{\alpha-1}. \tag{4.10.1}$$

Proof. We have

$$\begin{aligned}
\Omega(\xi_1, \xi_2) &= \int \frac{(e^{i\xi_1 y} - 1)(e^{i\xi_2 y} - 1)}{y|y|^\alpha}\, dy \\
&= \int \frac{e^{i(\xi_1+\xi_2)y} - e^{i\xi_1 y} - e^{i\xi_2 y} + 1}{y|y|^\alpha}\, dy
\end{aligned}$$

We first prove the result in the case $\alpha > 1$. We can see that

$$\Omega(\xi_1, \xi_2) = \lambda(\xi_1 + \xi_2) - \lambda(\xi_1) - \lambda(\xi_2),$$

where

$$\begin{aligned}
\lambda(\eta) &= \int \frac{e^{i\eta y} - i\eta y - 1}{y|y|^\alpha}\, dy \\
&= \eta|\eta|^{\alpha-1} \int \frac{e^{iy} - iy - 1}{y|y|^\alpha}\, dy = \eta|\eta|^{\alpha-1}\lambda(1)
\end{aligned}$$

Let $\lambda(1) := d(\alpha)$. In particular,

$$\lambda'(\eta) = \alpha d(\alpha) |\eta|^{\alpha-1}$$

We recall from Subsection 2.2 in [39] that

$$c(\alpha) = \int \frac{1 - \cos(y)}{|y|^\alpha}\, dy$$

On the other hand,

$$\begin{aligned}\lambda'(\eta) &= \int iy\frac{e^{i\eta y}-1}{y|y|^{\alpha-1}}\,dy = i\int \frac{e^{i\eta y}-1}{|y|^{\alpha-1}}\,dy\\ &= i|\eta|^{\alpha-1}\int \frac{e^{iy}-1}{|y|^{\alpha-1}}\,dy = -i|\eta|^{\alpha-1}c(\alpha)\end{aligned}$$

Now, this implies that $\alpha d(\alpha) = -ic(\alpha)$, hence

$$\lambda(\eta) = \eta|\eta|^{\alpha-1}d(\alpha) = -i\frac{c(\alpha)}{\alpha}\eta|\eta|^{\alpha-1} = -\frac{\omega(\eta)}{\alpha},$$

which implies the desired identity for Ω.

We now analyze the case $\alpha \in (0,1)$. We recall from (3.14) in [74] that

$$c(\alpha) = -2\int_0^\infty \frac{\cos(y)}{y^\alpha}\,dy = -\int \frac{\cos(y)}{|y|^\alpha}\,dy.$$

We can see that

$$\Omega(\xi_1,\xi_2) = \lambda(\xi_1+\xi_2) - \lambda(\xi_1) - \lambda(\xi_2),$$

where

$$\lambda(\eta) = \int \frac{e^{i\eta y}-1}{y|y|^\alpha}\,dy = \eta|\eta|^{\alpha-1}\int \frac{e^{iy}-1}{y|y|^\alpha}\,dy = \eta|\eta|^{\alpha-1}\lambda(1).$$

We denote $\lambda(1) := d(\alpha)$. In particular,

$$\lambda'(\eta) = \alpha d(\alpha)|\eta|^{\alpha-1}.$$

On the other hand,

$$\begin{aligned}\lambda'(\eta) &= \int iy\frac{e^{i\eta y}}{y|y|^{\alpha}}\,dy = i\int \frac{e^{i\eta y}}{|y|^{\alpha}}\,dy\\ &= i|\eta|^{\alpha-1}\int \frac{e^{iy}}{|y|^{\alpha}}\,dy = -i|\eta|^{\alpha-1}c(\alpha)\end{aligned}$$

Now, this again implies that $\alpha d(\alpha) = -ic(\alpha)$, hence

$$\lambda(\eta) = \eta|\eta|^{\alpha-1}d(\alpha) = -i\frac{c(\alpha)}{\alpha}\eta|\eta|^{\alpha-1} = -\frac{\omega(\eta)}{\alpha},$$

which implies the desired identity for Ω.

$\square$

This null structure is crucial for the normal form analysis, which we carry out in the next section.

We make the above discussion relating Q to Ω precise in the following lemma. Recall that we denote

$$\psi := \partial_x^{-1} F(\varphi_x).$$

Lemma 4.11. We have

$$Q(\varphi, v) = \Omega(\psi, v) + R(x, D)v$$

where

$$\|(\partial_x R)(x, D)v\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}.$$

Proof. We write

$$\begin{aligned} R(x, D)v &= Q(\varphi, v) - \Omega(\psi, v) \\ &= \int \frac{1}{|y|^{\alpha-1}} \frac{F(\delta^y \varphi) - \delta^y \partial_x^{-1} F(\varphi_x)}{|y|} \cdot (v(x+y) - v(x))\, dy \end{aligned} \tag{4.11.1}$$

which has symbol

$$r(x, \xi) = -\int \frac{1}{|y|^{\alpha-1}} \frac{F(\delta^y \varphi) - \delta^y \partial_x^{-1} F(\varphi_x)}{|y|} (e^{i\xi y} - 1)\, dy.$$

Then we have

$$\begin{aligned} (\partial_x R)(x, D)v &= \int \frac{1}{|y|^{\alpha-1}} \frac{F'(\delta^y \varphi)\delta^y \varphi_x - \delta^y F(\varphi_x)}{|y|} \cdot (v(x+y) - v(x))\, dy \\ &=: \int K(x, y) \cdot (v(x+y) - v(x))\, dy. \end{aligned}$$

We first estimate K, which we may write as

$$\begin{aligned} |y|^\alpha K(x, y) &= \frac{1}{y}(F'(b)(a-b) - (F(a) - F(b)) + \frac{1}{y}(F'(\delta^y \varphi) - F'(\varphi_x))(a-b) \\ &=: |y|^\alpha K_1(x, y) + |y|^\alpha K_2(x, y), \end{aligned}$$

where $a = \varphi_x(x+y)$, $b = \varphi_x(x)$. From K_1 we obtain a Taylor expansion,

$$\|K_1(\cdot, y)\|_{L^\infty_x} \lesssim_A \frac{1}{|y|^{\alpha-1}} \left\| \frac{a-b}{y} \right\|^2_{L^\infty_x} = \frac{1}{|y|^{\alpha-1}} \|\delta^y \varphi_x\|^2_{L^\infty_x}.$$

For K_2, we have

$$\begin{aligned} \|K_2(\cdot, y)\|_{L^\infty_x} &\lesssim_A \frac{1}{|y|^{\alpha-1}} \left\| \frac{\varphi(x+y) - \varphi(x) - y\varphi_x(x)}{y^2} \right\|_{L^\infty_x} \left\| \frac{a-b}{y} \right\|_{L^\infty_x} \\ &= \frac{1}{|y|^{\alpha-1}} \|\delta^{y,(2)} \varphi\|_{L^\infty_x} \|\delta^y \varphi_x\|_{L^\infty_x}, \end{aligned}$$

where $\delta^{y,(2)}$ denotes the second-order difference quotient.

By Minkowski's inequality,

$$\begin{aligned}\|(\partial_x R)(x,D)v\|_{L^2} &\lesssim \int \|K(\cdot,y)\|_{L^\infty_x}\|v(\cdot+y)-v(\cdot)\|_{L^2_x}\,dy \\ &\lesssim_A \int \frac{1}{|y|^{\alpha-1}}\|\delta^y\varphi_x\|_{L^\infty_x}(\|\delta^y\varphi_x\|_{L^\infty_x}+\|\delta^{y,(2)}\varphi\|_{L^\infty_x})\|v\|_{L^2_x}\,dy.\end{aligned}$$

By Lemma 4.8, we conclude

$$\|(\partial_x R)(x,D)v\|_{L^2} \lesssim_A B^2\|v\|_{L^2}.$$

□

4.11.1 The paradifferential flow

We consider the linearized equation (4.1.10), to which we associate its corresponding inhomogeneous paradifferential flow,

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v - \partial_x Q_{lh}(\varphi, v) = f, \tag{4.11.2}$$

where the frequency decomposition of the (essentially) quadratic form has been expressed as

$$\begin{aligned}Q(\varphi,v) &= \int \frac{1}{|y|^{\alpha-1}}T_{F(\delta^y\varphi)}|\delta|^y v\,dy + \int \frac{1}{|y|^{\alpha-1}}T_{|\delta|^y v}F(\delta^y\varphi)\,dy + \int \frac{1}{|y|^{\alpha-1}}\Pi(|\delta|^y v, F(\delta^y\varphi))\,dy \\ &=: Q_{lh}(\varphi,v)+Q_{hl}(\varphi,v)+Q_{hh}(\varphi,v).\end{aligned} \tag{4.11.3}$$

We analogously decompose $\Omega = \Omega_{lh} + \Omega_{hl} + \Omega_{hh}$.

In Section 4.15, we prove that the linearized equation (4.1.10) reduces to its paradifferential version (4.11.2), where f is perturbative in the sense that it satisfies balanced, cubic estimates. However, since Q and hence its paradifferential components $Q_{hl}(\varphi, v)$ and $Q_{hh}(\varphi, v)$ are essentially quadratic, we will first have to perform an appropriate paradifferential normal form change of variables for this to become apparent.

Here, we prove a preliminary quadratic estimate for the reduction, which will be used when constructing and evaluating the contributions of the normal form transformation later in Section 4.15.

We first extract the principal components of Ω_{lh}, which include a transport term of order 0 and a dispersive term of order $\alpha - 1$:

Lemma 4.12. We can express

$$\alpha Q_{lh}(\varphi,v) = c(\alpha)(T_{|D_x|^{\alpha-1}\partial_x\psi+\frac{\alpha}{c(\alpha)}R}v - T_{\partial_x\psi}|D_x|^{\alpha-1}v + \partial_x[T_\psi, |D_x|^{\alpha-1}]v). \tag{4.12.1}$$

Further, we have

$$Q_{lh}(\varphi, v) = c(\alpha)(T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi + \frac{\alpha}{c(\alpha)}R)}v - T_{\partial_x\psi}|D_x|^{\alpha-1}v) + \Gamma(\partial_x^2\psi, |D_x|^{\alpha-2}v) \tag{4.12.2}$$

where

$$\||D_x|^{\frac{2-\alpha}{2}}\Gamma\|_{L^2} \lesssim_A B\|v\|_{L^2}, \qquad \|\partial_x\Gamma\|_{L^2} \lesssim_A B\||D|^{\frac{\alpha}{2}}v\|_{L^2}, \tag{4.12.3}$$

as well the pointwise estimates

$$\||D_x|^{\frac{2-\alpha}{2}}\Gamma\|_{L^\infty} \lesssim_A B\|v\|_{L^\infty}, \qquad \|\partial_x\Gamma\|_{L^\infty} \lesssim_A B\||D|^{\frac{\alpha}{2}}v\|_{L^\infty}. \tag{4.12.4}$$

Proof. We use the resonance identity (4.10.1) to expand

$$\begin{aligned}\alpha\Omega_{lh}(\psi, v) &= c(\alpha)(T_{|D_x|^{\alpha-1}\partial_x\psi}v + [T_\psi, |D_x|^{\alpha-1}\partial_x]v) \\ &= c(\alpha)(T_{|D_x|^{\alpha-1}\partial_x\psi}v - T_{\partial_x\psi}|D_x|^{\alpha-1}v + \partial_x[T_\psi, |D_x|^{\alpha-1}]v).\end{aligned} \tag{4.12.5}$$

Combined with the low-high component of Lemma 4.11, which takes the form

$$Q_{lh}(\varphi, v) = \Omega_{lh}(\psi, v) + T_R v,$$

we obtain (4.12.1). To then obtain (4.12.2), the remaining commutator may be expressed as

$$\partial_x[T_\psi, |D_x|^{\alpha-1}]v = -c(\alpha)(\alpha-1)T_{\partial_x\psi}|D_x|^{\alpha-1}v + \alpha\Gamma(\partial_x^2\psi, |D_x|^{\alpha-2}v)$$

where Γ denotes the subprincipal remainder, which has a favorable balance of derivatives on the low frequency and thus may be estimated as (4.12.3) and (4.12.4). □

Proposition 4.13. Consider a solution v to (4.1.10). Then v satisfies

$$(\partial_t - c(\alpha)(T_{J^{-1}}|D_x|^{\alpha-1} - T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi + R)})\partial_x)v = f \tag{4.13.1}$$

where

$$\||D|^{-\frac{2-\alpha}{2}}f\|_{L^2} \lesssim_A B\|v\|_{L^2}. \tag{4.13.2}$$

Proof. We express (4.1.10) in terms of the paradifferential equation (4.11.2) with source,

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v - \partial_x Q_{lh}(\varphi, v) = \partial_x Q_{hl}(\varphi, v) + \partial_x Q_{hh}(\varphi, v).$$

We estimate the source terms. We see directly from definition that $\partial_x Q_{hl}(\varphi, v)$ has a favorable balance of derivatives, in the sense that φ occupies the position of the high frequency in the quadratic form, and thus when we estimate this term, we may allocate any positive fraction of the outer ∂_x derivative onto φ. In particular, we may choose to allocate a total of $\alpha/2 < 1$ derivatives onto φ to use the B control norm, so that

$$\||D|^{-\frac{2-\alpha}{2}}\partial_x Q_{hl}(\varphi, v)\|_{L^2} \lesssim_A B\|v\|_{L^2}.$$

We see that $\partial_x Q_{hl}(\varphi, v)$ satisfies (4.13.2) and may be absorbed into f. The balanced Q_{hh} term satisfies (4.13.2) in precisely the same way, so we have thus reduced (4.1.10) to (4.11.2).

It then suffices to apply (4.12.2) of Lemma 4.12 to the remaining paradifferential Q_{lh} term on the left hand side of (4.11.2) to obtain (4.13.1):

$$Q_{lh}(\varphi, v) = c(\alpha)(T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi+R)}v - T_{\partial_x\psi}|D_x|^{\alpha-1}v) + \Gamma(\partial_x^2\psi, |D_x|^{\alpha-2}v).$$

Here, by the lemma, the Γ contribution may be absorbed into f directly. Further, in (4.13.1), we have commuted the ∂_x outside Q_{lh} through the low frequency paracoefficients, since the cases where this derivative falls on the low frequency coefficients,

$$c(\alpha)(T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x^2\psi+\partial_x R)}v - T_{\partial_x^2\psi}|D_x|^{\alpha-1}v),$$

have a favorable balance of derivatives, satisfying (4.13.2).

□

4.13.1 Nonlinear equations

The paradifferential equation (4.13.1) will also be used in the context of the nonlinear solutions φ. To conclude this section, we prove preliminary quadratic bounds on the inhomogeneity of the paradifferential flow, in analogy with the preceding Proposition 4.13 for its linearized counterpart.

Proposition 4.14. Consider a solution φ to (4.1.2).

a) The solution φ satisfies

$$(\partial_t - c(\alpha)(T_{J^{-1}}|D_x|^{\alpha-1} - T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi+R)})\partial_x)\varphi = f \tag{4.14.1}$$

where

$$\||D_x|^{\frac{2-\alpha}{2}}f\|_{L^\infty} \lesssim_A B, \qquad \|\partial_x f\|_{L^\infty} \lesssim_A B^2. \tag{4.14.2}$$

b) The same holds for ψ in the place of φ.

Proof. We first consider the case of φ. We paradifferentially decompose $Q(\varphi, \partial_x\varphi)$ in (4.1.2) to write it in terms of the paradifferential equation (4.11.2) with source,

$$\partial_t\varphi - c(\alpha)|D_x|^{\alpha-1}\partial_x\varphi - Q_{lh}(\varphi, \partial_x\varphi) = Q_{hl}(\varphi, \partial_x\varphi) + Q_{hh}(\varphi, \partial_x\varphi).$$

As with the linearized equation, we estimate the source terms. We see directly from definition that $Q_{hl}(\varphi, \partial_x\varphi)$ has a favorable balance of derivatives in the sense that the copy of φ which occupies the position of the low frequency already has a derivative on it, and thus when we estimate this term, we may retain any positive fraction of this derivative on the low frequency φ, displacing the remainder onto the high frequency copy of φ. In particular, we may choose to allocate a total of $\alpha/2 < 1$ derivatives onto this φ to use the B control norm, so that

$$\||D|^{\frac{2-\alpha}{2}} Q_{hl}(\varphi, \partial_x \varphi)\|_{L^\infty} \lesssim_A B, \quad \|\partial_x Q_{hl}(\varphi, \partial_x \varphi)\|_{L^\infty} \lesssim B^2.$$

We see that $Q_{hl}(\varphi, \partial_x\varphi)$ satisfies (4.14.2) and may be absorbed into f. The balanced Q_{hh} term satisfies (4.14.2) in precisely the same way, so we have thus replaced $Q(\varphi, \partial_x\varphi)$ in (4.1.2) with $Q_{lh}(\varphi, \partial_x\varphi)$. In turn, it then suffices to apply Lemma 4.12 to obtain (4.14.1).

In preparation to prove the same result for ψ, we next reduce the equation for $T_{F'(\varphi_x)}\varphi$ to that of φ. It suffices to apply the paracoefficient $T_{F'(\varphi_x)}$ to (4.14.1), and estimate the commutators. This is straightforward for the spatial paradifferential terms, applying Lemma 4.3 and observing a favorable balance of derivatives.

For the time derivative, we substitute (4.14.1) for the time derivative of φ:

$$T_{F''(\varphi_x)\partial_x\partial_t\varphi}\varphi = T_{F''(\varphi_x)\left(c(\alpha)\partial_x T_{J-1}|D_x|^{\alpha-1}\partial_x\varphi + \frac{c(\alpha)}{\alpha}\partial_x T_{|D_x|^{\alpha-1}\partial_x\psi + R}\partial_x\varphi + \partial_x f\right)}\varphi.$$

The estimate (4.14.2) on $\partial_x f$ in the paracoefficient implies that its contribution in this context also satisfies (4.14.2). For the remaining terms, the favorable balance of derivatives, with two or more derivatives on the low frequency paracoefficient, again implies that we may absorb their contribution into f.

To conclude the proof for ψ, it suffices to apply the Moser estimate of Theorem 4.4.1, other than for the time derivative, for which we need to estimate

$$\partial_x^{-1}(F'(\varphi_x)\partial_x\partial_t\varphi) - \partial_t T_{F'(\varphi_x)}\varphi.$$

We decompose this into

$$[\partial_x^{-1}, T_{F'(\varphi_x)}]\partial_x\partial_t\varphi$$

which we estimate directly, using the favorable balance of derivatives, and

$$\partial_x^{-1}T_{\partial_x\partial_t\varphi}F'(\varphi_x) + \partial_x^{-1}\Pi(\partial_x\partial_t\varphi, F'(\varphi_x))$$

which is similar to the time derivative commutation in the previous reduction.

□

4.15 Reduction to the paradifferential equation

In this section, we reduce energy estimates and well-posedness for the linearized equation (4.1.10) to that of the inhomogeneous paradifferential flow (4.11.2),

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v - \partial_x Q_{lh}(\varphi, v) = f.$$

For the energy estimates to be balanced, we in turn require that the inhomogeneity f satisfy balanced cubic estimates.

Unfortunately, the paradifferential errors $Q_{hl}(\varphi, v)$ and $Q_{hh}(\varphi, v)$ are essentially quadratic rather than cubic, and in particular do not satisfy balanced cubic estimates. On the other hand, as we noted in (4.9.1), Q is the quadratic form associated to the resonance function for the dispersion relation of (4.1.2), up to leading order and a low frequency coefficient, which suggests that a normal form transformation can be effective in reducing our equation to one with a cubic nonlinearity. The classical normal form transformation associated to this nonlinearity is

$$\tilde{v} = v - \frac{1}{\alpha}\partial_x(\psi v), \qquad \psi = \partial_x^{-1} F(\varphi_x). \tag{4.15.1}$$

Unfortunately, (4.15.1) has two drawbacks:

1. It is unbounded, and hence we cannot apply it directly, and
2. quartic (essentially cubic) error terms arising in the equation for $\tilde{v}$ given by (4.15.1) are still unbalanced.

To address the first issue, we instead consider only the bounded paradifferential components of (3.11.1),

$$\tilde{v} = v - \frac{1}{\alpha}\partial_x(T_v\psi) - \frac{1}{\alpha}\partial_x \Pi(v, \psi), \tag{4.15.2}$$

which is well-suited to the purpose of reducing the problem to the paradifferential equation (4.11.2). To address the second issue, we augment (4.15.2) with a low frequency Jacobian coefficient which eliminates the quartic and higher order residuals:

$$\tilde{v} = v - \frac{1}{\alpha}\partial_x T_{T_J v}\psi - \frac{1}{\alpha}\partial_x \Pi(T_J v, \psi), \qquad J = (1 - \partial_x \psi)^{-1}. \tag{4.15.3}$$

Proposition 4.16. Consider a solution v to (4.1.10). Then we have

$$\partial_t \tilde{v} - c(\alpha)|D_x|^{\alpha-1}\partial_x \tilde{v} - \partial_x Q_{lh}(\varphi, \tilde{v}) = \tilde{f}, \tag{4.16.1}$$

where $\tilde{f}$ satisfies balanced cubic estimates,

$$\|\tilde{f}\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}. \tag{4.16.2}$$

Proof. We express v satisfying (4.1.10) in terms of the paradifferential equation (4.11.2) with source,

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v - \partial_x Q_{lh}(\varphi, v) = \partial_x Q_{hl}(\varphi, v) + \partial_x Q_{hh}(\varphi, v).$$

Unlike in Proposition 4.13, we do not estimate the source terms directly, which would not suffice to obtain an error estimate like (3.12.2). Instead, it suffices to establish the following cancellation with the contribution from the normal form correction (4.15.3),

$$\partial_t \partial_x T_{T_J v}\psi - c(\alpha)|D_x|^{\alpha-1}\partial_x^2 T_{T_J v}\psi - \partial_x Q_{lh}(\varphi, \partial_x T_{T_J v}\psi) = \alpha \partial_x Q_{hl}(\varphi, v) + \tilde{f}, \tag{4.16.3}$$

with the analogous relationship for the balanced Π component of the correction, with Q_{hh}.

To show (4.16.3), we first observe that using Lemma 4.12, we may replace $\partial_x Q_{lh}$ on the left hand side of (4.16.3) by its principal components. The Γ error is estimated using the second estimate of (4.12.3),

$$\|\partial_x \Gamma\|_{L^2} \lesssim_A B \||D|^{\frac{\alpha}{2}} \partial_x T_{T_J v} \psi\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}$$

and may be absorbed into $\tilde{f}$. It thus suffices to show

$$\left(\partial_t - c(\alpha)\partial_x (T_{J^{-1}}|D_x|^{\alpha-1} - T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi + R)})\right) \partial_x T_{T_J v}\psi = \alpha \partial_x Q_{hl}(\varphi, v) + \tilde{f}. \tag{4.16.4}$$

We begin the proof of (4.16.4) by computing the time derivative. The case where ∂_t falls on the low frequency J may be absorbed into $\tilde{f}$ due to a favorable balance of derivatives, in the sense that $1+\alpha$ derivatives have fallen on the lowest frequency term and can be allocated to higher frequencies as necessary to obtain the desired $\tilde{f}$ estimate. More precisely, we use (4.14.1) to write

$$\partial_t J = J^2 \partial_x \partial_t \psi = c(\alpha) J^2 \partial_x (T_{J^{-1}}|D_x|^{\alpha-1} - T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi + R)})\partial_x \psi + J^2 \partial_x f$$

so that we can we can estimate for instance the contribution of the first term,

$$\|\partial_x T_{T_{J^2 \partial_x^2 |D_x|^{\alpha-1}\psi} v} \psi\|_{L^2} \lesssim_A B^2 \|v\|_{L^2},$$

with similar estimates for the other contributions, and using the first estimate of (4.14.2) for the contribution of f.

In the remaining cases, ∂_t falls on the middle frequency v or the high frequency ψ, so we use (4.13.1) and (4.14.1) respectively to write

$$\begin{aligned} &\partial_x T_{T_J \partial_t v}\psi = c(\alpha)\partial_x T_{T_J\left(T_{J^{-1}}|D_x|^{\alpha-1}\partial_x v + \frac{1}{\alpha} T_{|D_x|^{\alpha-1}\partial_x\psi + R}\partial_x v + f\right)}\psi, \\ &\partial_x T_{T_J v}\partial_t \psi = c(\alpha)\partial_x T_{T_J v}\left(T_{J^{-1}}|D_x|^{\alpha-1}\partial_x\psi + T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi + R)}\partial_x\psi + f_\psi\right). \end{aligned} \tag{4.16.5}$$

We consider the first, second, and third contributions from the two equations of (4.16.5) in pairs:

i) The first terms in (4.16.5) combine with the second term on the left in (3.12.4),

$$c(\alpha)\partial_x (T_{T_J T_{J^{-1}}|D_x|^{\alpha-1}\partial_x v}\psi + T_{T_J v} T_{J^{-1}}|D_x|^{\alpha-1}\partial_x \psi - T_{J^{-1}}|D_x|^{\alpha-1}\partial_x T_{T_J v}\psi), \tag{4.16.6}$$

to form $\alpha \partial_x Q_{hl}(\varphi, v)$, modulo balanced errors which may be absorbed into $\tilde{f}$. To see this, we will use in each of the three terms that $J^{-1}J = 1$. As we do so, we need to take care that any paraproduct errors and commutators yield balanced errors.

First, we observe that in the third term of (4.16.6), we can apply the commutator estimate

$$\|\partial_x[T_{J^{-1}}, |D_x|^{\alpha-1}\partial_x]T_{T_J v}\psi\|_{L^2} \lesssim_A B^2\|v\|_{L^2}.$$

Then using Lemma 4.4 to compose paraproducts in each of the three terms of (4.16.6), we have

$$c(\alpha)\partial_x(T_{|D_x|^{\alpha-1}\partial_x v}\psi + T_{J^{-1}T_J v}|D_x|^{\alpha-1}\partial_x\psi - |D_x|^{\alpha-1}\partial_x T_{J^{-1}T_J v}\psi).$$

For the latter two terms, we will also use Lemma 4.4 to compose paraproducts, before applying $J^{-1}J = 1$. To do so, we first need to exchange multiplication by J^{-1} with the paraproduct $T_{J^{-1}}$. However, the error from this exchange is not directly perturbative. Instead, we perform the exchange for the two terms simultaneously, to observe a cancellation in the form of the commutator

$$c(\alpha)\partial_x(T_{T_{T_J v}J^{-1}}|D_x|^{\alpha-1}\partial_x\psi - |D_x|^{\alpha-1}\partial_x T_{T_{T_J v}J^{-1}}\psi),$$

which has a favorable balance of derivatives and may be absorbed into $\tilde{f}$. The same holds for the analogous cases with $\Pi(J^{-1}, T_J v)$. We have thus reduced (4.16.6) to

$$c(\alpha)\partial_x(T_{|D_x|^{\alpha-1}\partial_x v}\psi + T_v|D_x|^{\alpha-1}\partial_x\psi - |D_x|^{\alpha-1}\partial_x T_v\psi) = \alpha\partial_x\Omega_{hl}(\psi, v)$$

which by Lemma 4.11 coincides with $\alpha\partial_x Q_{hl}(\varphi, v)$ up to balanced errors, as desired.

ii) The second terms in (4.16.5),

$$\frac{c(\alpha)}{\alpha}\partial_x(T_{T_J T_{|D_x|^{\alpha-1}\partial_x\psi+R}\partial_x v}\psi + T_{T_J v}T_{|D_x|^{\alpha-1}\partial_x\psi+R}\partial_x\psi), \tag{4.16.7}$$

combine to cancel the third term on the left hand side of (4.16.4), up to balanced errors. To see this, we apply the commutator Lemma 4.3 to exchange the first term of (4.16.7) with

$$\frac{c(\alpha)}{\alpha}\partial_x T_{T_{|D_x|^{\alpha-1}\partial_x\psi+R}T_J\partial_x v}\psi.$$

We can freely exchange the low frequency paraproduct $T_{|D_x|^{\alpha-1}\partial_x\psi+R}$ with a standard product, since

$$\|\partial_x T_{T_{T_J\partial_x v}|D_x|^{\alpha-1}\partial_x\psi + T_{T_J\partial_x v}R}\psi\|_{L^2} \lesssim_A B^2\|v\|_{L^2} \tag{4.16.8}$$

and likewise for the balanced Π case. We thus have

$$\frac{c(\alpha)}{\alpha}\partial_x T_{T_J\partial_x v\cdot|D_x|^{\alpha-1}\partial_x\psi+R}\psi.$$

Then applying Lemma 4.4 for splitting paraproducts, and returning to (4.16.7), we arrive at

$$\frac{c(\alpha)}{\alpha}\partial_x(T_{T_J\partial_x v}T_{|D_x|^{\alpha-1}\partial_x\psi+R}\psi + T_{T_J v}T_{|D_x|^{\alpha-1}\partial_x\psi+R}\partial_x\psi).$$

Lastly, we factor out a derivative,

$$\frac{c(\alpha)}{\alpha}\partial_x^2 T_{T_J v} T_{|D_x|^{\alpha-1}\partial_x\psi+R}\psi$$

where we have absorbed the cases where the factored derivative falls on J or $|D_x|^{\alpha-1}\partial_x\psi + R$ into $\tilde{f}$, similar to (4.16.8). After one more instance of the commutator Lemma 4.3, we arrive at the third term on the left hand side of (4.16.4) as desired.

iii) By Propositions 4.13 and 4.14 respectively, the contributions from f and f_ψ satisfy (4.16.2) and may be absorbed into $\tilde{f}$.

□

We also prove a similar but easier balanced estimate for the reduction of the nonlinear equation to the paradifferential flow, in the $\dot{H}^s$ setting. In this situation, the normal form correction only consists of a balanced Π component:

$$\tilde{\varphi} = \varphi - \frac{1}{\alpha}\Pi(\psi, T_J\partial_x\varphi). \tag{4.16.9}$$

Proposition 4.17. Consider a solution φ to (4.1.2). Then we have

$$\partial_t\tilde{\varphi} - 2c(\alpha)|D_x|^{\alpha-1}\partial_x\tilde{\varphi} - \partial_x Q_{lh}(\varphi, \tilde{\varphi}) = \tilde{f}, \tag{4.17.1}$$

where $\tilde{f}$ satisfies balanced cubic estimates,

$$\|\tilde{f}\|_{\dot{H}^s} \lesssim_A B^2\|\varphi\|_{\dot{H}^s_x}. \tag{4.17.2}$$

Proof. First observe that we have

$$\partial_t\varphi - 2c(\alpha)|D_x|^{\alpha-1}\partial_x\varphi - \partial_x Q_{lh}(\varphi, \varphi) = Q_{hh}(\varphi, \partial_x\varphi).$$

Then the normal form analysis is similar to the analysis for the balanced paradifferential error of the linear equation in Proposition 4.16,

$$\partial_x Q_{hh}(\varphi, v). \tag{4.17.3}$$

To see this, first observe that the computations correspond precisely at the algebraic level. In particular, here the derivative ∂_x falls on one of the inputs of Q_{hh}, but accordingly the derivative has been shifted to the input of Π in the normal form correction (4.16.9), preserving the correspondence with the computations for the linear equation in Proposition 4.16.

To see that we can obtain balanced estimates in the $\dot{H}^s$ setting rather than just the L^2 setting, first observe that in each of the estimates in the proof of Proposition 4.16, we can easily obtain at least one B from the estimate of the low frequency variable. Then since we are in the balanced Π setting where we may shift derivatives freely between the inputs, we can obtain a second B, with s outstanding derivatives, which can then be placed on the remaining high frequency factor.

□

4.18 Energy estimates for the paradifferential equation

In this section we obtain energy estimates for the paradifferential flow (4.11.2). We define the modified energy

$$E(v) := \int v \cdot T_{\tilde{J}(\psi_x)} v \, dx, \qquad \tilde{J}(x) = J(x)^{-\frac{1}{\alpha}}.$$

Unlike in the SQG case, where $\alpha = 1$, the energy functional here is in general a fully nonlinear function of ψ_x. In particular, observe that the paradifferential normal form used in the previous section is a linearization of the normal form underlying the modified energy used here.

Indeed, the normal form of the previous section would suggest that here we use the transformation

$$\tilde{v} = v - \frac{1}{\alpha} \partial_x T_\psi v,$$

corresponding to a cubic modified energy of the form

$$\int \tilde{v}^2 - \frac{2}{\alpha} \tilde{v} \cdot \partial_x T_\psi \tilde{v} \, dx = \int \tilde{v} \cdot T_{1-\frac{1}{\alpha}\psi_x} \tilde{v} \, dx,$$

which only provides the first order approximation of the correct energy $E(v)$.

Proposition 4.19. Consider a solution v to (4.11.2). We have

$$\frac{d}{dt} E(v) \lesssim_A B^2 \|v\|_{L^2}^2 + \|f\|_{L^2} \|v\|_{L^2}. \tag{4.19.1}$$

Proof. Without loss of generality we consider the homogeneous case of (4.11.2), assuming $f = 0$. Using (4.11.2) and (4.14.1) respectively to expand time derivatives, we have

$$\begin{aligned} \frac{\alpha}{c(\alpha)} \frac{d}{dt} \int v \cdot T_{\tilde{J}} v \, dx &= 2 \frac{\alpha}{c(\alpha)} \int \partial_t v \cdot T_{\tilde{J}} v \, dx - \frac{\alpha}{c(\alpha)} \int v \cdot T_{\frac{1}{\alpha} J \tilde{J} \cdot \partial_x \partial_t \psi} v \, dx \\ &= 2 \frac{\alpha}{c(\alpha)} \int (c(\alpha) |D_x|^{\alpha-1} \partial_x v + \partial_x Q_{lh}(\varphi, v)) \cdot T_{\tilde{J}} v \, dx \\ &\quad - \int v \cdot T_{J \tilde{J} \cdot \partial_x (T_{J^{-1}} |D_x|^{\alpha-1} \partial_x \psi + \frac{1}{\alpha} T_{|D_x|^{\alpha-1} \partial_x \psi + R} \partial_x \psi + f)} v \, dx. \end{aligned} \tag{4.19.2}$$

The contribution from f may be estimated using (4.14.2) and discarded.

We expand the contribution from Q_{lh} in the second term on the right hand side of (4.19.2) using (4.12.1) of Lemma 4.12,

$$2 \int \partial_x (T_{|D_x|^{\alpha-1} \partial_x \psi + R} v + T_\psi |D_x|^{\alpha-1} \partial_x v - |D_x|^{\alpha-1} \partial_x T_\psi v) \cdot T_{\tilde{J}} v \, dx. \tag{4.19.3}$$

For clarity we also collect the other remaining terms from the right hand side of (4.19.2),

$$\int 2\alpha|D_x|^{\alpha-1}\partial_x v\cdot T_{\tilde{J}}v\,dx - \int v\cdot T_{J\tilde{J}\cdot\partial_x(T_{J-1}|D_x|^{\alpha-1}\partial_x\psi+\frac{1}{\alpha}T_{|D_x|^{\alpha-1}\partial_x\psi+R}\partial_x\psi)}v\,dx. \tag{4.19.4}$$

We observe cancellations between (4.19.3) and (4.19.4) in the following three steps, while ensuring that all residues are bounded by the right hand side of (3.15.1).

i) From the first term in (4.19.3), we have after cyclically integrating by parts,

$$\int v\cdot T_{\partial_x^2|D_x|^{\alpha-1}\psi+\partial_x R}T_{\tilde{J}}v - v\cdot T_{|D_x|^{\alpha-1}\partial_x\psi+R}T_{\partial_x\tilde{J}}v + v\cdot[T_{\tilde{J}},T_{|D_x|^{\alpha-1}\partial_x\psi+R}]\partial_x v\,dx.$$

The commutator satisfies (4.19.1) by Lemma 4.3. On the other hand, the first two terms cancel with the second integral in (4.19.4), up to errors also satisfying (4.19.1). To see these cancellations, we first observe that

$$JJ^{-1}=1, \qquad \partial_x\tilde{J} = -\frac{1}{\alpha}J\tilde{J}\partial_x^2\psi.$$

These identities, which relate literal products, do not appear directly in (4.19.3) and (4.19.4), which instead use several instances of paraproducts. However, this technicality can by bypassed by using Lemma 4.4 to compose paraproducts, and observing that any contribution with two or more derivatives on the lowest frequency has a favorable balance of derivatives and satisfies (4.19.1). For instance, from the second term of the second integrand in (4.19.4), we have the perturbative component

$$\left|\int v\cdot T_{T_{\partial_x^2|D_x|^{\alpha-1}\psi}\partial_x\psi}v\,dx\right| \lesssim_A B^2\|v\|_{L^2}^2.$$

ii) We consider the remaining two terms of (4.19.3), which may be viewed as a (skew symmetric) commutator, plus a residual integral:

$$2\int \partial_x[T_\psi,|D_x|^{\alpha-1}]\partial_x v\cdot T_{\tilde{J}}v\,dx - 2\int\partial_x|D_x|^{\alpha-1}T_{\partial_x\psi}v\cdot T_{\tilde{J}}v\,dx. \tag{4.19.5}$$

In this step, we would like to replace the commutator with its principal component,

$$-2(\alpha-1)\int T_{\partial_x\psi}|D_x|^{\alpha-1}\partial_x v\cdot T_{\tilde{J}}v\,dx.$$

However, since this principal component is not directly skew symmetric, we also include an additional correction:

$$(\alpha-1)\int[T_{\partial_x\psi},|D_x|^{\alpha-1}\partial_x]v\cdot T_{\tilde{J}}v\,dx.$$

After peeling away the principal component and the correction from the commutator, the residual is skew-symmetric:

$$L(\partial_x^2\psi, |D_x|^{\alpha-1}(\cdot)) := \partial_x[T_\psi, |D_x|^{\alpha-1}]\partial_x + (\alpha-1)(T_{\partial_x\psi}|D_x|^{\alpha-1}\partial_x - \frac{1}{2}[T_{\partial_x\psi}, |D_x|^{\alpha-1}\partial_x]).$$

We thus have a commutator form for its contribution,

$$\int T_{\tilde{J}} L(\partial_x^2\psi, |D_x|^{\alpha-1}v)\cdot v\,dx = \frac{1}{2}\int (T_{\tilde{J}}L(\partial_x^2\psi, |D_x|^{\alpha-1}v) - L(\partial_x^2\psi, |D_x|^{\alpha-1}T_{\tilde{J}}v))\cdot v\,dx$$

for which we have the desired balanced estimate. Collecting the principal component, the correction, and the second integral from (4.19.5), we have

$$-\int \big((\alpha+1)|D_x|^{\alpha-1}\partial_x T_{\partial_x\psi}v + (\alpha-1)T_{\partial_x\psi}|D_x|^{\alpha-1}\partial_x v\big)\cdot T_{\tilde{J}}v\,dx.$$

Together with the first term of (4.19.4), it remains to consider

$$\int \big((\alpha+1)|D_x|^{\alpha-1}\partial_x T_{J^{-1}}v + (\alpha-1)T_{J^{-1}}|D_x|^{\alpha-1}\partial_x v\big)\cdot T_{\tilde{J}}v\,dx. \tag{4.19.6}$$

iii) From the unit-coefficient terms in (4.19.6), we have the commutator

$$\int [|D_x|^{\alpha-1}\partial_x, T_{J^{-1}}]v\cdot T_{\tilde{J}}v\,dx =: \int L(\partial_x^2\psi, |D_x|^{\alpha-1}v)\cdot T_{\tilde{J}}v\,dx. \tag{4.19.7}$$

On the other hand, from the α-coefficient terms, we integrate by parts on the first term, and commute the paracoefficients on the second term using Lemma 4.3,

$$\alpha\int -T_{J^{-1}}v\cdot|D_x|^{\alpha-1}\partial_x T_{\tilde{J}}v + T_{\tilde{J}}|D_x|^{\alpha-1}\partial_x v\cdot T_{J^{-1}}v\,dx,$$

thus also obtaining a commutator,

$$-\alpha\int [|D_x|^{\alpha-1}\partial_x, T_{\tilde{J}}]v\cdot T_{J^{-1}}v\,dx = \int L(\alpha\partial_x\tilde{J}, |D_x|^{\alpha-1}v)\cdot T_{J^{-1}}v\,dx.$$

Then since

$$\alpha\partial_x\tilde{J} = -J\tilde{J}\partial_x^2\psi,$$

we may apply Lemma 4.5 to reduce to

$$-\int T_{J\tilde{J}}L(\partial_x^2\psi, |D_x|^{\alpha-1}v)\cdot T_{J^{-1}}v\,dx.$$

Using self-adjointness with Lemma 4.4 and $JJ^{-1} = 1$, we have cancellation with the previous commutator (4.19.7). Precisely, to split and recombine the J and $\tilde{J}$ terms while maintaining balanced errors, derivatives are shifted from the low frequency $\partial_x^2\psi$ to v, before being absorbed using Lemma 4.4 on the J and $\tilde{J}$ paraproducts.

$\square$

Proposition 4.20. Assume that $A \ll 1$ and $B \in L^2_t$. Then there exists an energy functional $E_{lin}(v)$ such that we have the following:

a) Norm equivalence:

$$E_{lin}(v) \approx_A \|v\|^2_{L^2_x}$$

b) Energy estimates:

$$\frac{d}{dt}E_{lin}(v) \lesssim_A B^2\|v\|^2_{L^2_x}$$

Remark 4.20.1. We note that the linearized equation (4.1.10) is well-posed in L^2_x. We do not need this result, but we briefly discuss the main steps in its proof. The main idea is to prove a similar estimate for the adjoint equation, interpreted as a backward evolution in the space L^2. More precisely, the adjoint equation corresponding to the linearized equation has the form

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v - Q(\varphi, \partial_x v) = 0.$$

By carrying out a paradifferential normal form transformation, similar to the one from the proof of Proposition 4.16, this can be reduced to

$$\partial_t v - c(\alpha)|D_x|^{\alpha-1}\partial_x v - Q_{lh}(\varphi, \partial_x v) = 0.$$

By considering the modified energy functional

$$\int v \cdot T_{J^{\frac{1}{\alpha}}} v \, dx,$$

we obtain the desired energy estimate for the dual problem in a manner similar to Proposition 4.19. We can now infer existence for the solutions to the linearized equation (4.1.10) by a standard duality argument (for the general theory, see Theorem 23.1.2 in [67]).

Proof. Let $E_{lin}(v) = E(\tilde{v})$, where $E(\cdot)$ is defined in Proposition 4.19, and $\tilde{v}$ is defined in Proposition 4.16. Part a) is immediate, whereas part b) follows from Proposition 4.19. $\square$

4.21 Higher order energy estimates

In this section we obtain higher order energy estimates. As the commutators with D^s are quadratic and thus not perturbative, they pose an additional obstacle. To address this, we first eliminate the highest order terms by using a Jacobian exponential conjugation, at the cost of generating additional terms, but at lower order. We then carry out a normal form analysis in order to eliminate the remaining non-perturbative terms.

Proposition 4.22. Let $s \geq 0$. Given v solving (3.7.2), there exists a normalized variable v^s such that

$$\partial_t v^s - c(\alpha)|D_x|^{\alpha-1}\partial_x v^s - \partial_x Q_{lh}(\varphi, v^s) = f + \mathscr{R}(\varphi, v),$$

with

$$\|v^s - |D_x|^s v\|_{L^2_x} \lesssim_A A\|v\|_{\dot{H}^s_x}$$

and $\mathscr{R}(\varphi, v)$ satisfying balanced cubic estimates,

$$\|\mathscr{R}(\varphi, v)\|_{L^2} \lesssim_A B^2\|v\|_{L^2}. \tag{4.22.1}$$

Proof. Let v satisfy (4.11.2), where without loss of generality, $f = 0$. The natural approach is to reduce the equation for $v^s := |D_x|^s v$ to (4.11.2) with a perturbative inhomogeneity. However, the commutators arising from such a reduction are quadratic, and cannot satisfy balanced cubic estimates. In particular, they cannot be seen as directly perturbative. We will address these errors via a conjugation combined with a normal form correction.

In preparation, we use Lemmas 4.11 and 4.12 to rewrite Q_{lh} in (4.11.2). Denoting

$$\mathscr{P} = T_{1-\frac{1}{\alpha}\partial_x\psi}|D_x|^{\alpha-1} + T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi+R)} + \partial_x[T_{\frac{1}{\alpha}\psi}, |D_x|^{\alpha-1}],$$

we obtain

$$\partial_t v - c(\alpha)\partial_x\mathscr{P}v = 0. \tag{4.22.2}$$

Then $v^s := |D_x|^s v$ satisfies the same equation, but with the following commutators in the source:

$$\begin{aligned}&\frac{c(\alpha)}{\alpha}\partial_x([|D_x|^s, T_{-\partial_x\psi}]|D_x|^{\alpha-1} + [|D_x|^s, T_{|D_x|^{\alpha-1}\partial_x\psi+R}] + \partial_x[|D_x|^s, [T_\psi, |D_x|^{\alpha-1}]])v \\ &\quad =: L(\partial_x^2\psi, |D_x|^{\alpha-1}v^s) - L(|D_x|^{\alpha-1}\partial_x^2\psi, v^s) + c(\alpha)\partial_x^2[|D_x|^s, [T_{\frac{1}{\alpha}\psi}, |D_x|^{\alpha-1}]]v + \mathscr{R}\end{aligned} \tag{4.22.3}$$

where we have absorbed $\partial_x R$ into $\mathscr{R}$ and L denotes an order zero paradifferential bilinear form,

$$L(\partial_x f, u) = -\frac{c(\alpha)}{\alpha}\partial_x[|D_x|^s, T_f]|D_x|^{-s}u. \tag{4.22.4}$$

In particular, observe that the principal term of L is given by

$$L(g, u) \approx -\frac{c(\alpha)}{\alpha}sT_g u.$$

To address the two L contributions, which are quadratic and not directly perturbative, we apply two steps:

a) We first apply a conjugation to v^s which improves the leading order of the contributions of the L terms from $\alpha - 1$ and 0 to $\alpha - 2$ and -1 respectively.

b) We then apply a normal form transformation yielding cubic, balanced source terms.

a) We begin by computing the equation for the conjugated variable

$$\tilde{v}^s := T_{J^{-\frac{s}{\alpha}}} v^s.$$

To do so, it suffices to apply $T_{J^{-\frac{s}{\alpha}}}$ to (3.18.3) and consider the commutators. These will include a ∂_t commutator, a ∂_x commutator, and a $|D_x|^{\alpha-1}$ commutator.

i) First, we use (4.14.1) to expand the ∂_t commutator,

$$\begin{aligned}
[T_{J^{-\frac{s}{\alpha}}}, \partial_t] v^s &= -\frac{s}{\alpha} T_{J^{1-\frac{s}{\alpha}} \partial_x \partial_t \psi} v^s \\
&= -\frac{s}{\alpha} T_{J^{1-\frac{s}{\alpha}} \partial_x \left(c(\alpha) T_{J^{-1}|D_x|^{\alpha-1}\partial_x \psi} + \frac{c(\alpha)}{\alpha} T_{|D_x|^{\alpha-1}\partial_x \psi + R} \partial_x \psi + f \right)} v^s.
\end{aligned}$$

Here the contribution from f may be estimated using (4.14.2) and discarded. Further, distributing the outer ∂_x, and due to a favorable balance of derivatives when ∂_x falls on the lowest frequency variables, we may absorb these cases into $\mathscr{R}$ we reduce to

$$-\frac{sc(\alpha)}{\alpha} T_{J^{-\frac{s}{\alpha}} \left(T_J T_{J^{-1}|D_x|^{\alpha-1}\partial_x^2 \psi} + \frac{1}{\alpha} T_J T_{|D_x|^{\alpha-1}\partial_x \psi + R} \partial_x^2 \psi \right)} v^s.$$

Lastly, we apply Lemma 4.4 to merge and split paraproducts, reducing to

$$-\frac{sc(\alpha)}{\alpha} T_{|D_x|^{\alpha-1}\partial_x^2 \psi + \frac{1}{\alpha} T_{J|D_x|^{\alpha-1}\partial_x \psi + R} \partial_x^2 \psi} \tilde{v}^s. \tag{4.22.5}$$

Observe that the first part of (4.22.5) cancels with the principal term of the second L on the right hand side of (4.22.3). The second part of (4.22.5) will cancel with a contribution from ii) below.

ii) Next, we consider the commutator of $T_{J^{-\frac{s}{\alpha}}}$ with the outer ∂_x. We obtain

$$\frac{sc(\alpha)}{\alpha} T_{J^{1-\frac{s}{\alpha}} \partial_x^2 \psi} \mathscr{P} v^s.$$

Here it is convenient to apply (4.12.2) of Lemma 4.12 to write this as

$$\frac{sc(\alpha)}{\alpha} T_{J^{1-\frac{s}{\alpha}} \partial_x^2 \psi} \left((T_{J^{-1}} |D_x|^{\alpha-1} + T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x \psi + R)}) v^s + \Gamma(\partial_x^2 \psi, |D_x|^{\alpha-1} \partial_x^{-1} v^s) \right).$$

We first apply Lemma 4.5 to split $J^{1-\frac{s}{\alpha}}$ from $\partial_x^2 \psi$ in the outer paraproduct. Then applying Lemma 4.4 along with 4.3, to compose and commute paraproducts, this reduces, modulo perturbative terms, to

$$\frac{sc(\alpha)}{\alpha} T_{\partial_x^2 \psi} |D_x|^{\alpha-1} \tilde{v}^s + \frac{sc(\alpha)}{\alpha^2} T_{J|D_x|^{\alpha-1}\partial_x \psi \cdot \partial_x^2 \psi + \partial_x^2 \psi \cdot R} \tilde{v}^s.$$

The first term above cancels with the principal term of the first L. The second term cancels with the remaining part of the ∂_t commutator above in (4.22.5). To see this cancellation, we have freely exchanged multiplication by $J|D_x|^{\alpha-1}\partial_x\psi$ with a paraproduct, as the difference has a favorable balance of derivatives and is thus perturbative.

iii) Returning to the commutator of $T_{J^{-\frac{s}{\alpha}}}$ with the dispersive term, it remains to consider the commutator with the inner $|D_x|^{\alpha-1}$, where we have used Lemma 4.3 to discard any paraproduct commutators. We have

$$-c(\alpha)\partial_x T_{J^{-1}}[T_{J^{-\frac{s}{\alpha}}}, |D_x|^{\alpha-1}]v^s$$

whose principal term $\frac{s}{\alpha}c(\alpha)(\alpha-1)T_{\partial_x^2\psi}|D_x|^{\alpha-1}\tilde{v}^s$ cancels with the principal term of the double commutator on the right hand side of (4.22.3).

To conclude, we have

$$\begin{aligned}
&\partial_t\tilde{v}^s - c(\alpha)\partial_x\mathscr{P}\tilde{v}^s\\
&\quad= (L(\partial_x^2\psi, |D_x|^{\alpha-1}\tilde{v}^s) + \frac{sc(\alpha)}{\alpha}T_{\partial_x^2\psi}|D_x|^{\alpha-1}\tilde{v}^s)\\
&\quad\quad - (L(|D_x|^{\alpha-1}\partial_x^2\psi, \tilde{v}^s) + \frac{sc(\alpha)}{\alpha}T_{|D_x|^{\alpha-1}\partial_x^2\psi}\tilde{v}^s)\\
&\quad\quad + \frac{c(\alpha)}{\alpha}T_{J^{-\frac{s}{\alpha}}}\partial_x^2[|D_x|^s, [T_\psi, |D_x|^{\alpha-1}]]v - c(\alpha)\partial_x T_{J^{-1}}[T_{J^{-\frac{s}{\alpha}}}, |D_x|^{\alpha-1}]v^s + f\\
&\quad=: L_0(\partial_x^3\psi, |D_x|^{\alpha-1}\partial_x^{-1}\tilde{v}^s) - L_0(|D_x|^{\alpha-1}\partial_x^3\psi, \partial_x^{-1}\tilde{v}^s) + L_1(\partial_x^3\psi, |D_x|^{\alpha-1}\partial_x^{-1}\tilde{v}^s) + f
\end{aligned} \tag{4.22.6}$$

where f satisfies (4.22.1). Here L_0 and L_1 denote order zero paradifferential bilinear forms, respectively

$$\begin{aligned}
L_0(\partial_x^2 f, |D_x|^{\alpha-1}\partial_x^{-1}u) &= L(\partial_x f, u) + \frac{sc(\alpha)}{\alpha}T_{\partial_x f}u,\\
L_1(\partial_x^2 f, |D_x|^{\alpha-1}\partial_x^{-1}T_{J^{-\frac{s}{\alpha}}}u) &= \frac{c(\alpha)}{\alpha}T_{J^{-\frac{s}{\alpha}}}\partial_x^2[|D_x|^s, [T_{\partial_x^{-1}f}, |D_x|^{\alpha-1}]]|D_x|^{-s}u\\
&\quad - c(\alpha)\partial_x T_{J^{-1}}[T_{J^{-\frac{s}{\alpha}}}, |D_x|^{\alpha-1}]u.
\end{aligned}$$

Observe that since L_i are all order 0 paradifferential bilinear forms, we have reduced the terms of the inhomogeneity to order -1.

b) We next choose a normal form transformation to reduce the quadratic components of the inhomogeneity to balanced cubic terms. Let

$$\tilde{w}^s = \frac{1}{c(\alpha)\alpha}T_J L_0(\partial_x^2\psi, \partial_x^{-1}\tilde{v}^s).$$

Then we claim that $\tilde{u}^s := \tilde{v}^s - \tilde{w}^s$ is the desired normal form transform. To see this, it remains to compute $(\partial_t - c(\alpha)\partial_x \mathscr{P})\tilde{w}^s$, which may be expressed using Lemma 4.12 as

$$\left(\partial_t - c(\alpha)\partial_x(T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi+R)} - T_{J^{-1}}|D_x|^{\alpha-1})\right)\tilde{w}^s + \partial_x\Gamma(\partial_x^2\psi, |D_x|^{\alpha-2}\tilde{w}^s), \tag{4.22.7}$$

and observe cancellation with the three L_i bilinear forms on the right hand side of (4.22.6). To see this, we partition the computation into the following subgroups:

i) When the full equation of (4.22.7) falls on the high frequency $\tilde{v}^s$ input of L_0, we may use (4.22.6) to see that the contribution has a favorable balance of derivatives and may be absorbed into f.

ii) We may commute the equation freely with the low frequency J due to a favorable balance of derivatives, absorbing the contribution again into f.

It remains to consider commutators of the terms of the equation (4.22.7) across the low frequency $\partial_x^2\psi$ input of $\tilde{w}^s$.

iii) We first consider the commutators involving the operators

$$c(\alpha)T_{\frac{1}{\alpha}(|D_x|^{\alpha-1}\partial_x\psi+R)}\partial_x + \Gamma(\partial_x^2\psi, |D_x|^{\alpha-1}\partial_x^{-1}(\cdot)) \circ \partial_x.$$

of (4.22.7), where we have freely commuted the ∂_x forward with balanced errors. The contribution from the Γ term may also be absorbed into f due to a favorable balance. The remaining contribution

$$\frac{1}{\alpha^2}T_J L_0(T_{|D_x|^{\alpha-1}\partial_x\psi+R}\partial_x^3\psi, \partial_x^{-1}\tilde{v}^s)$$

will cancel with a contribution of step iv) below.

iv) For the case when ∂_t falls on the low frequency input of L_0, we apply equation (4.14.1). Precisely, the two non-perturbative contributions on the left hand side of (4.14.1) cancel respectively with the second L_0 source term in (4.22.6), and the remaining contribution of step iii) above.

v) From the remaining dispersive term $c(\alpha)T_{J^{-1}}\partial_x|D_x|^{\alpha-1}$ of (4.22.7), the case when ∂_x falls on the low frequency input of L_0 while $|D_x|^{\alpha-1}$ has commuted to the high frequency input cancels with the first L_0 source term in (4.22.6).

vi) From the same term $c(\alpha)T_{J^{-1}}\partial_x|D_x|^{\alpha-1}$, it remains to consider the commutators with $|D_x|^{\alpha-1}$, where the ∂_x remains in front.

We first apply Lemma 4.5 to split $J^{1-\frac{s}{\alpha}}$ from $\partial_x^2\psi$ in the outer paraproduct, together with Lemma 4.4 with $JJ^{-1} = 1$, and opening the definition of L_0, we have

$$\frac{c(\alpha)}{\alpha}\partial_x^2[|D_x|^{\alpha-1}, [|D_x|^s, T_\psi]]|D_x|^{-s}\tilde{v}^s - \frac{sc(\alpha)}{\alpha}\partial_x[|D_x|^{\alpha-1}, T_{\partial_x\psi}]\tilde{v}^s.$$

We claim that these two terms cancel with the two terms of L_1 respectively. Indeed, for the first term, we commute using Lemma 4.3 to reduce to (suppressing a factor of $c(\alpha)$ from all terms henceforth)

$$\frac{1}{\alpha}T_{J^{-\frac{s}{\alpha}}}\partial_x^2[|D_x|^{\alpha-1},[|D_x|^s,T_\psi]]v$$

which cancels with the double commutator term of L_1. For the second term, we also commute using Lemma 4.3 to reduce to

$$\frac{s}{\alpha}T_{J^{-\frac{s}{\alpha}}}\partial_x[T_{\partial_x\psi},|D_x|^{\alpha-1}]v^s =: -\frac{s(\alpha-1)}{\alpha}T_{J^{-\frac{s}{\alpha}}}T_{\partial_x^2\psi}|D_x|^{\alpha-1}v^s + T_{J^{-\frac{s}{\alpha}}}L_2(\partial_x^3\psi,|D_x|^{\alpha-1}\partial_x^{-1}v).$$

On the other hand, the second term of L_1 may be expressed as

$$-\partial_x T_{J^{-1}}[T_{J^{-\frac{s}{\alpha}}},|D_x|^{\alpha-1}]v^s = \frac{s(\alpha-1)}{\alpha}T_{J^{-1}}T_{J^{1-\frac{s}{\alpha}}\partial_x^2\psi}v^s - T_{J^{-1}}L_2(J^{1-\frac{s}{\alpha}}\partial_x^3\psi,|D_x|^{\alpha-1}\partial_x^{-1}v).$$

We apply Lemma 4.5 to split $J^{1-\frac{s}{\alpha}}$ into its own paraproduct. Then these terms cancel, up to an application of Lemma 4.4. Precisely, we move one derivative from $\partial_x^2\psi$ onto the highest frequency, after which we apply Lemma 4.4 with $JJ^{-1}=1$. □

We thus obtain the following:

Proposition 4.23. Assume that $A \ll 1$ and $B \in L_t^2$. Let $s \geq 0$. Then there exist energy functionals $E^{(s)}(v)$ such that we have the following:

a) Norm equivalence:

$$E^{(s)}(v) \approx_A \|v\|_{\dot{H}_x^s}^2$$

b) Energy estimates:

$$\frac{d}{dt}E^{(s)}(v) \lesssim_A B^2\|v\|_{\dot{H}_x^s}^2$$

Proof. Let $E^{(s)}(v) = E(v^s)$, where $E(\cdot)$ is defined in Proposition 4.19, and v^s is defined in Proposition 4.22. Part a) is immediate, whereas part b) follows from Proposition 4.19. □

4.24 Local well-posedness

In this section we prove Theorem 4.1.1, which is our main local well-posedness result. We follow the general approach outlined in introductory primer [87]. We consider $\varphi_0 \in (\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})$, with $s_1 < \frac{3}{2}$, $s_2 > \frac{\alpha+3}{2}$. Let $\varphi_0^h = (\varphi_0)_{\leq h}$, where $h \in \mathbb{Z}$. Since $\varphi_0^h \to \varphi_0$ in $(\dot{H}_x^{s_1} \cap \dot{H}_x^{s_2})$, we may assume that $\|\varphi_0^h\|_{(\dot{H}_x^{s_1}\cap\dot{H}_x^{s_2})} < R$ for all h.

We construct a uniform $(\dot{H}_x^{s_1}\cap\dot{H}_x^{s_2})$ frequency envelope $\{c_k\}_{k\in\mathbb{Z}}$ for φ_0 having the following properties:

a) Uniform bounds:
$$\|P_k(\varphi_0^h)\|_{\dot{H}_x^{s_1}\cap \dot{H}_x^{s_2}} \lesssim c_k,$$

b) High frequency bounds:
$$\|\varphi_0^h\|_{\dot{H}_x^{s_1}\cap \dot{H}_x^{N}} \lesssim 2^{h(N-s_2)}c_h, \qquad N > s_2,$$

c) Difference bounds:
$$\|\varphi_0^{h+1}-\varphi_0^h\|_{L_x^2} \lesssim 2^{-s_2 h}c_h,$$

d) Limit as $h\to\infty$:
$$\varphi_0^h \to \varphi_0 \in \dot{H}_x^{s_1}\cap \dot{H}_x^{s_2}.$$

Let φ^h be the solutions with initial data φ_0^h, whose existence is guaranteed instance by [6]. Using the energy estimate for the solution φ of (4.1.2) from Proposition 4.23 and Proposition 4.17, we deduce that there exists $T = T(\|\varphi_0\|_{H_x^s}) > 0$ on which all of these solutions are defined, with high frequency bounds

$$\|\varphi^h\|_{C_t^0(\dot{H}_x^{s_1}\cap \dot{H}_x^{N})} \lesssim \|\varphi_0^h\|_{(\dot{H}_x^{s_1}\cap \dot{H}_x^{N})} \lesssim 2^{h(N-s_2)}c_h.$$

Further, by using the energy estimates for the solution of the linearized equation from Proposition 4.20, we have
$$\|\varphi^{h+1}-\varphi^h\|_{C_t^0 L_x^2} \lesssim 2^{-s_2 h}c_h.$$
By interpolation, we infer that

$$\|\varphi^{h+1}-\varphi^h\|_{C_t^0(\dot{H}_x^{s_1}\cap \dot{H}_x^{s_2})} \lesssim c_h.$$

As in [87], we get
$$\|P_k\varphi^h\|_{C_t^0(\dot{H}_x^{s_1}\cap \dot{H}_x^{s_2})} \lesssim c_k$$
and that
$$\|\varphi^{h+k}-\varphi^h\|_{C_t^0(\dot{H}_x^{s_1}\cap \dot{H}_x^{s_2})} \lesssim c_{h\leq\cdot<h+k} = \left(\sum_{n=h}^{h+k-1} c_n^2\right)^{\frac{1}{2}}$$
for every $k\geq 1$. Thus, φ^h converges to an element φ belonging to $C_t^0(\dot{H}_x^{s_1}\cap \dot{H}_x^{s_2})([0,T]\times\mathbb{R})$. Moreover, we also obtain

$$\|\varphi^h-\varphi\|_{C_t^0(\dot{H}_x^{s_1}\cap \dot{H}_x^{s_2})} \lesssim c_{\geq h} = \left(\sum_{n=h}^{\infty} c_n^2\right)^{\frac{1}{2}}. \tag{4.24.1}$$

We now prove continuity with respect to the initial data. We consider a sequence

$$\varphi_{0j} \to \varphi_0 \in \dot{H}_x^{s_1}\cap \dot{H}_x^{s_2}$$

and an associated sequence of H^s_x-frequency envelopes $\{c^j_k\}_{k\in\mathbb{Z}}$, each satisfying the analogous properties enumerated above for c_k, and further such that $c^j_k \to c_k$ in $l^2(\mathbb{Z})$. In particular,

$$\|\varphi^h_j - \varphi_j\|_{C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)} \lesssim c^j_{\geq h} = \left(\sum_{n=h}^{\infty}(c^j_n)^2\right)^{\frac{1}{2}}. \tag{4.24.2}$$

Using the triangle inequality with (4.24.1) and (4.24.2), we write

$$\begin{aligned}\|\varphi_j - \varphi\|_{C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)} &\lesssim \|\varphi^h - \varphi\|_{C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)} + \|\varphi^h_j - \varphi_j\|_{C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)} + \|\varphi^h_j - \varphi^h\|_{C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)} \\ &\lesssim c_{\geq h} + c^j_{\geq h} + \|\varphi^h_j - \varphi^h\|_{C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)}.\end{aligned}$$

To address the third term, we observe that for every fixed h, $\varphi^h_j \to \varphi^h$ in $\dot{H}^{s_1}_x \cap \dot{H}^{s_2}_x$. We conclude $\varphi_j \to \varphi$ in $C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)([0,T]\times\mathbb{R})$ and therefore $\varphi_j \to \varphi$ in $C^0_t(\dot{H}^{s_1}_x\cap\dot{H}^{s_2}_x)([0,T]\times\mathbb{R})$.

4.25 Global well-posedness

In this section we prove the global well-posedness part of Theorem 4.1.2 and the dispersive bounds of the resulting solution, by using the wave packet testing method of Ifrim-Tataru. This approach is systematically presented in [86].

4.25.1 Notation

Consider the linear flow

$$i\partial_t\varphi - A(D)\varphi = 0$$

and the linear operator

$$L = x - tA'(D).$$

In our setting, we have the symbol

$$a(\xi) = -c(\alpha)\xi|\xi|^{\alpha-1}$$

and thus

$$A(D) = -c(\alpha)D|D|^{\alpha-1}, \qquad L = x + t\alpha c(\alpha)|D|^{\alpha-1}.$$

We define the weighted energy space ($s_0 < \min\{1,\alpha\}$, $s > \alpha+2$)

$$\|\varphi\|_X = \|\varphi\|_{\dot{H}^{s_0}\cap\dot{H}^s} + \|L\partial_x\varphi\|_{L^2}.$$

We also define the pointwise control norm

$$\|\varphi\|_Y = \||D_x|^{1-\delta}\varphi\|_{L^\infty_x} + \||D_x|^{\frac{\alpha}{2}+\delta}\varphi_x\|_{L^\infty_x}.$$

We partition the frequency space into dyadic intervals I_λ localized at dyadic frequencies $\lambda \in 2^{\mathbb{Z}}$, and consider the associated partition of velocities

$$J_\lambda = a'(I_\lambda)$$

which form a covering of $(-\infty, 0)$. For each λ, $|J_\lambda| \approx \lambda^{\alpha-1}$. To these intervals J_λ we select reference points $v_\lambda \in J_\lambda$, and consider an associated spatial partition of unity

$$1 = \sum_\lambda \chi_\lambda(x), \qquad \text{supp } \chi_\lambda \subseteq \overline{J_\lambda}, \qquad \chi_\lambda = 1 \text{ on } J_\lambda,$$

where $\overline{J_\lambda}$ is a slight enlargement of J_λ, of comparable length, uniformly in λ.

Lastly, we consider the related spatial intervals, tJ_λ, with reference points $x_\lambda = tv_\lambda \in tJ_\lambda$.

4.25.2 Overview of the proof

We provide a brief overview of the proof.

1. We make the bootstrap assumption for the pointwise bound

$$\|\varphi(t)\|_Y \lesssim C\epsilon \langle t\rangle^{-\frac{1}{2}} \tag{4.25.1}$$

where C is a large constant, in a time interval $t \in [0, T]$ where $T > 1$.

2. The energy estimates for (4.1.2) and the linearized equation will imply

$$\|\varphi(t)\|_X \lesssim \langle t\rangle^{C^2\epsilon^2} \|\varphi(0)\|_X. \tag{4.25.2}$$

3. We aim to improve the bootstrap estimate (4.25.1) to

$$\|\varphi(t)\|_Y \lesssim \epsilon \langle t\rangle^{-\frac{1}{2}}. \tag{4.25.3}$$

We use vector field inequalities to derive bounds of the form

$$\|\varphi(t)\|_Y \lesssim \epsilon \langle t\rangle^{-\frac{1}{2}+C\epsilon^2}, \tag{4.25.4}$$

which is the desired bound but with an extra $t^{C\epsilon^2}$ loss.

4. In order to rectify the extra loss, we use the wave packet testing method. Namely, we define a suitable asymptotic profile γ, which is then shown to be an approximate solution for an ordinary differential equation. This enables us to obtain suitable bounds for the asymptotic profile without the aforementioned loss, which can then be transferred back to the solution φ.

4.25.3 Energy estimates

From Proposition 4.23 and Grönwall's lemma, together with the fact that $\epsilon \ll 1$, we get that

$$\|\varphi(t,x)\|_{H^s_x} \lesssim e^{C\int_0^t C(A(\tau))B(\tau)^2\,d\tau}\|\varphi_0\|_{H^s_x}.$$

Let $u = L\partial_x\varphi + t\alpha\int |y|^{1-\alpha}F(\delta^y\varphi)|\delta|^y\varphi_x\,dy$, which satisfies the linearized equation with error $-\int |y|^{1-\alpha}F'(\delta^y\varphi)\delta^y\varphi|\delta|^y\varphi_x\,dy$, which is clearly balanced.

Indeed, we start with the family of rescaled solutions

$$\varphi^\kappa = \kappa^{-1}\varphi(\kappa^\alpha t, \kappa x)$$

We have

$$\frac{d}{d\kappa}\varphi^\kappa = -\kappa^{-2}\varphi(\kappa^\alpha t,\kappa x) + \kappa^{-1}\varphi_t(\kappa^\alpha t,\kappa x)\alpha\kappa^{\alpha-1}t + \kappa^{-1}\varphi_x(\kappa^\alpha t,\kappa x)\kappa x,$$

which are all solutions of the linearized equation.

By taking $\kappa = 1$, we get

$$\begin{aligned}
\frac{d}{d\kappa}_{|\kappa=1}\varphi^\kappa(t,x) &= -\varphi(t,x) + t\alpha\left(c(\alpha)|D_x|^{\alpha-1}\varphi_x + \int |y|^{1-\alpha}F(\delta^y\varphi))|\delta|^y\varphi_x\,dy\right) + x\varphi_x(t,x)\\
&= -\varphi(t,x) + L\varphi_x + t\alpha\int |y|^{1-\alpha}F(\delta^y\varphi)|\delta|^y\varphi_x\,dy\\
&= u(t,x) - \varphi(t,x),
\end{aligned}$$

We may now write

$$\begin{aligned}
0 &= (\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})(u-\varphi) - \partial_x Q(\varphi, u-\varphi)\\
&= (\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})u - \partial_x Q(\varphi,u) - (\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})\varphi + \partial_x Q(\varphi,\varphi)\\
&= (\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})u - \partial_x Q(\varphi,u) - (\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})\varphi + Q(\varphi,\varphi_x)\\
&+ \int |y|^{1-\alpha}F'(\delta^y\varphi)\delta^y\varphi|\delta|^y\varphi_x\,dy\\
&= (\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})u - \partial_x Q(\varphi,u) + \int |y|^{1-\alpha}F'(\delta^y\varphi)\delta^y\varphi|\delta|^y\varphi_x\,dy,
\end{aligned}$$

hence

$$(\partial_t - c(\alpha)\partial_x|D_x|^{\alpha-1})u - \partial_x Q(\varphi,u) = -\int |y|^{1-\alpha}F'(\delta^y\varphi)\delta^y\varphi|\delta|^y\varphi_x\,dy.$$

Lemma 4.6 implies that

$$\begin{aligned}
&\left\|\int |y|^{1-\alpha}F'(\delta^y\varphi)\delta^y\varphi|\delta|^y\varphi_x\,dy\right\|_{L^2_x} + \left\|\int |y|^{1-\alpha}F(\delta^y\varphi)|\delta|^y\varphi_x\,dy\right\|_{L^2_x}\\
&\qquad\lesssim \|\varphi_x\|^2_{L^\infty_x}(\||D_x|^{\alpha-\delta}\varphi\|_{L^2_x} + \|\varphi_x\|_{L^2_x}),
\end{aligned}$$

which shows that our error in the linearized equation is balanced, and that

$$\left\| t\alpha \int |y|^{1-\alpha} F(\delta^y \varphi)|\delta|^y \varphi_x \right\|_{L^2_x} \lesssim C^2\varepsilon^2(\||D_x|^{\alpha-\delta}\|_{L^2_x} + \|\varphi_x\|_{L^2_x})$$

From Proposition 4.20, along with Grönwall's lemma and the fact that $\epsilon \ll 1$, we have

$$\|L\partial_x\varphi(t,x)\|_{L^2_x} \lesssim e^{C\int_0^t C(A(\tau))B(\tau)^2\, d\tau}\|u_0\|_{L^2_x}.$$

Along with the bootstrap assumptions, these readily imply that

$$\|\varphi\|_X \lesssim \|\varphi(t)\|_{\dot{H}^{s_0}_x\cap \dot{H}^s_x} + \|L\partial_x\varphi(t)\|_{L^2_x} \lesssim \epsilon e^{C^2\epsilon^2\int_0^t\langle s\rangle^{-1}\, ds} \lesssim \epsilon\langle t\rangle^{C^2\epsilon^2}. \tag{4.25.5}$$

4.25.4 Vector field bounds

Proposition 2.1 from [86] implies that

$$\|\varphi_\lambda\|^2_{L^\infty_x} \lesssim \frac{1}{t\lambda^{\alpha-1}}(\|\varphi_\lambda\|_{L^2_x}\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|^2_{L^2_x}).$$

When $\lambda \leq 1$,

$$\begin{aligned}\||D_x|^{1-\delta}\varphi_\lambda\|_{L^\infty_x} &\lesssim \frac{1}{\sqrt{t}}\lambda^{\delta_1}(\|\lambda^{2-2\delta-2\delta_1}\varphi_\lambda\|^{1/2}_{L^2_x}\|L\partial_x\varphi_\lambda\|^{1/2}_{L^2_x} + \|\lambda^{1-\delta-\delta_1}\varphi_\lambda\|_{L^2_x})\\ &\lesssim \frac{1}{\sqrt{t}}\lambda^{\delta_1}\|\varphi\|_X\end{aligned}$$

and when $\lambda > 1$,

$$\||D_x|^{\frac{\alpha}{2}+\delta}\partial_x\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-\delta}(\|\lambda^{\alpha+2+4\delta}\varphi_\lambda\|^{1/2}_{L^2_x}\|L\partial_x\varphi_\lambda\|^{1/2}_{L^2_x} + \|\lambda^{\frac{\alpha}{2}+1+2\delta}\varphi_\lambda\|_{L^2_x}) \lesssim \frac{1}{\sqrt{t}}\lambda^{-\delta}\|\varphi\|_X.$$

By dyadic summation and Bernstein's inequality, we deduce the bound

$$\|\varphi\|_Y = \|\langle D_x\rangle^{2\delta+\frac{\alpha}{2}}|D_x|^{1-\delta}\varphi\|_{L^\infty_x} \lesssim \frac{\|\varphi\|_X}{\sqrt{t}}. \tag{4.25.6}$$

By the localized dispersive estimate [86, Proposition 5.1],

$$|\varphi_\lambda(x)|^2 \lesssim \frac{1}{|x-x_\lambda|t\lambda^{\alpha-2}}(\|L\varphi_\lambda\|_{L^2_x} + \lambda^{-1}\|\varphi_\lambda\|_{L^2_x})^2,$$

which implies that

$$\|(1-\chi_\lambda)\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{1}{2}-\alpha}}{t}(\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{\frac{1}{2}-\alpha}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \tag{4.25.7}$$

We also record the bound

$$\|(1-\chi_\lambda)\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{1}{2}-\alpha}}{t}(\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{\frac{1}{2}-\alpha}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}), \tag{4.25.8}$$

which follows directly from [86, Proposition 5.1].

To end this section we consider derivatives and difference quotients:

Lemma 4.26. We have

$$\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x} + \|(1-\chi_\lambda)\delta^y\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{3}{2}-\alpha}}{t}\|\varphi\|_X,$$

as well as the L^2_x-bounds

$$\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^2_x} + \|(1-\chi_\lambda)\delta^y\varphi_\lambda\|_{L^2_x} \lesssim \frac{\lambda^{1-\alpha}\|\varphi\|_X}{t}.$$

Proof. We use the bounds

$$|\partial_x(\chi_\lambda(x/t)| \lesssim t^{-1}\lambda^{1-\alpha}$$

Here, $\lambda^{1-\alpha}$ is the size of the interval J_λ. From (4.25.7) applied for $\partial_x\varphi$,

$$\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x} \lesssim \frac{1}{t}\lambda^{1-\alpha}\|\chi_\lambda'\varphi_\lambda\|_{L^\infty_x} + \|(1-\chi_\lambda)\partial_x\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{3}{2}-\alpha}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x})$$

For the bounds involving the difference quotient, from 4.25.7 applied for $\delta^y\varphi$, we have

$$\begin{aligned}
\|(1-\chi_\lambda)\delta^y\varphi_\lambda\|_{L^\infty_x} &\lesssim \frac{\lambda^{\frac{3}{2}-\alpha}}{t}(\|L\delta^y\varphi_\lambda\|_{L^2_x} + \lambda^{-1}\|\delta^y\varphi_\lambda\|_{L^2_x}) \\
&\lesssim \frac{\lambda^{\frac{3}{2}-\alpha}}{t}(\|\delta^y(L\varphi_\lambda)\|_{L^2_x} + \|\varphi_\lambda(x+y)\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \\
&\lesssim \frac{\lambda^{\frac{3}{2}-\alpha}}{t}(\|L\partial_x\varphi_\lambda\|_{L^2_x} + \|\varphi_\lambda\|_{L^2_x}) \\
&\lesssim \frac{\lambda^{\frac{3}{2}-\alpha}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x})
\end{aligned}$$

The other bounds are proved similarly. □

4.26.1 Wave packets

We construct wave packets as follows. Given the dispersion relation $a(\xi)$, the group velocity v satisfies

$$v = a'(\xi) = -c(\alpha)\alpha|\xi|^{\alpha-1},$$

so we denote

$$\xi_v = -\left(\frac{-v}{c(\alpha)\alpha}\right)^{\frac{1}{\alpha-1}}.$$

Then we define the linear wave packet $\mathbf{u}^v$ associated with velocity v by

$$\mathbf{u}^v = a''(\xi_v)^{-\frac{1}{2}}\chi(y)e^{it\phi(x/t)}, \qquad y = \frac{x-vt}{t^{\frac{1}{2}}a''(\xi_v)^{\frac{1}{2}}},$$

where the phase ϕ is given by

$$\phi(v) = v\xi_v - a(\xi_v),$$

and χ is a unit bump function, such that $\int \chi(y)\,dy = 1$.

In particular, when $v \in J_\lambda$, $|\phi'(v)| \approx \lambda$, and $|\phi'(v)| \approx \lambda^{2-\alpha}$.

We remark that we will typically use the frequency localization $\mathbf{u}^v_\lambda = P_\lambda \mathbf{u}^v$ with $v \in J_\lambda$.

We observe that since

$$\partial_v(|a''(\xi_v)|^{-\frac{1}{2}}) \simeq (a''(\xi_v)^{-\frac{1}{2}})^{\frac{4-3\alpha}{2-\alpha}},$$

we may write

$$\partial_v \mathbf{u}^v = -\tilde{L}\mathbf{u}^v + (a''(\xi_v)^{-\frac{1}{2}})^{\frac{2-2\alpha}{2-\alpha}}\mathbf{u}^{v,II} = t^{\frac{1}{2}}(a''(\xi_v)^{-\frac{1}{2}})\mathbf{u}^v + (a''(\xi_v)^{-\frac{1}{2}})^{\frac{2-2\alpha}{2-\alpha}}\mathbf{u}^{v,II} \tag{4.26.1}$$

where

$$\tilde{L} = t(\partial_x - i\phi'(x/t))$$

and $\mathbf{u}^{v,II}$ has a similar wave packet form. We also recall from [86, Lemmas 4.4, 5.10] the sense in which $\mathbf{u}^v$ is a good approximate solution:

Lemma 4.27. The wave packet $\mathbf{u}^v$ solves an equation of the form

$$(i\partial_t - A(D))\mathbf{u}^v = t^{-\frac{3}{2}}(L\mathbf{u}^{v,I} + \mathbf{r}^v)$$

where $\mathbf{u}^{v,I}, \mathbf{r}^v$ have wave packet form,

$$\mathbf{u}^{v,I} \approx a''(\xi_v)^{-\frac{1}{2}}\mathbf{u}^v, \qquad \mathbf{r}^v \approx \xi_v^{-1}a''(\xi_v)^{-\frac{1}{2}}\mathbf{u}^v.$$

The asymptotic profile at frequency λ is meaningful when the associated spatial region tJ_λ dominates the wave packet scale at frequency λ:

$$\delta x \approx t^{\frac{1}{2}}a''(\lambda)^{\frac{1}{2}} \lesssim |tJ_\lambda| \approx t\lambda a''(\lambda).$$

This corresponds to

$$t \gtrsim \lambda^{-2}a''(\lambda)^{-1} \approx \lambda^{-\alpha}.$$

Accordingly we define

$$\mathscr{D} = \{(t,v) \in \mathbb{R}^+ \times (-\infty, 0) : v \in J_\lambda,\ t \gtrsim \lambda^{-\alpha}\}.$$

4.27.1 Wave packet testing

In this section we establish estimates on the asymptotic profile function

$$\gamma^\lambda(t,v) := \langle \varphi, \mathbf{u}^v_\lambda \rangle_{L^2_x} = \langle \varphi_\lambda, \mathbf{u}^v \rangle_{L^2_x}.$$

We will see that γ^λ is essentially supported in the region $v \in J_\lambda$.

We will also use the following crude bounds involving the higher regularity of γ^λ:

Lemma 4.28. We have

$$\|\chi_\lambda \partial_v^n \gamma^\lambda\|_{L^\infty} \lesssim t^{\frac{1}{2}}(\lambda^{1-\alpha} + t^{\frac{1}{2}}\lambda^{1-\frac{\alpha}{2}})^n \|\varphi_\lambda\|_{L^\infty_x},$$
$$\|\chi_\lambda \partial_v^n \gamma^\lambda\|_{L^2} \lesssim (t\lambda^{2-\alpha})^{\frac{1}{4}}(\lambda^{1-\alpha} + t^{\frac{1}{2}}\lambda^{1-\frac{\alpha}{2}})^n \|\varphi_\lambda\|_{L^2_x},$$

and

$$\|\chi_\lambda \partial_v \gamma^\lambda\|_{L^\infty} \lesssim t^{\frac{1}{4}}\lambda^{-\frac{1}{2}-\frac{\alpha}{4}}(1+\lambda^{-1})\|\varphi\|_X + t^{\frac{1}{2}}\lambda^{1-\alpha}\|\varphi_\lambda\|_{L^\infty_x}.$$

Proof. Using the second form of $\partial_v \mathbf{u}^v$ in (4.26.1), we have

$$|\chi_\lambda \partial_v \gamma^\lambda| = |\chi_\lambda \langle \varphi_\lambda, \partial_v \mathbf{u}^v \rangle| \lesssim t^{\frac{1}{2}}(t^{\frac{1}{2}}\lambda^{1-\frac{\alpha}{2}} + \lambda^{1-\alpha})\|\varphi_\lambda\|_{L^\infty_x}$$

where the $t^{\frac{1}{2}}$ loss in front arises from the L^1 norm of the wave packet. Higher derivatives are obtained similarly, along with the L^2 estimates.

For the last bound, we use the first form of $\partial_v \mathbf{u}^v$ in (4.26.1). The contribution from the wave packet $\mathbf{u}^{v,II}$ is easily estimated as above. For the remaining bound, Lemma 2.3 from [86] implies that

$$|\langle \varphi_\lambda, \tilde{L}\mathbf{u}^v \rangle| \lesssim (t\lambda^{2-\alpha})^{\frac{1}{4}}\|\tilde{L}\varphi_\lambda\|_{L^2_x} \lesssim t^{\frac{1}{4}}\lambda^{-\frac{1}{2}-\frac{\alpha}{4}}\|\varphi_\lambda\|_X,$$

which finishes the proof. $\square$

Approximate profile

We recall from [86] that γ^λ provides a good approximation for the profile of φ. In our setting, we will also need to compare the profile with the differentiated flow $\partial_x \varphi$. Define

$$r^\lambda(t,x) = \varphi_\lambda(t,x) - t^{-\frac{1}{2}}\gamma^\lambda(t,x/t)e^{-it\phi(x/t)}.$$

Lemma 4.29. Let $t \geq 1$. Then we have

$$\|\chi_\lambda(x/t) r^\lambda\|_{L^\infty_x} \lesssim t^{-\frac{3}{4}}\lambda^{\frac{\alpha}{4}-\frac{1}{2}}\|\tilde{L}\varphi_\lambda\|_{L^2_x},$$
$$\|\chi_\lambda(x/t)\partial_v r^\lambda\|_{L^\infty_x} \lesssim t^{\frac{1}{4}}\lambda^{\frac{\alpha}{4}-\frac{1}{2}}\|\tilde{L}\partial_x\varphi_\lambda\|_{L^2_x} + (\lambda^{1-\alpha} + t^{\frac{1}{2}}\lambda^{1-\frac{\alpha}{2}})\|\varphi_\lambda\|_{L^\infty}.$$

Proof. The first estimate may be obtained from the proof of [86, Proposition 4.7]. For the latter, we use the first representation in (4.26.1) to write

$$e^{it\phi(v)}\partial_v(\gamma(t,v)e^{-it(\phi(v))}) = t\langle \partial_x\varphi_\lambda, \mathbf{u}^v\rangle + \langle \varphi_\lambda, it(\phi'(\cdot/t)-\phi'(v))\mathbf{u}^v\rangle + (a''(\xi_v)^{-\frac{1}{2}})^{\frac{2-2\alpha}{2-\alpha}}\langle \varphi_\lambda, \mathbf{u}^{v,II}\rangle. \tag{4.29.1}$$

Namely, we have used

$$\partial_v\mathbf{u}^v = -\tilde{L}\mathbf{u}^v + (a''(\xi_v)^{-\frac{1}{2}})^{\frac{2-2\alpha}{2-\alpha}}\mathbf{u}^{v,II}, \tag{4.29.2}$$

where $\tilde{L} = t(\partial_x - i\phi'(x/t))$.

To address the first term, we see that we may apply the undifferentiated estimate with $\partial_x\varphi_\lambda$ in place of φ_λ. Precisely, we may apply the first estimate on

$$\partial_x\varphi_\lambda(t,x) - t^{-\frac{1}{2}}\langle \partial_x\varphi_\lambda, \mathbf{u}^{x/t}\rangle e^{-it\phi(x/t)}.$$

We estimate the third term of (4.29.1) via

$$t^{-\frac{1}{2}}(a''(\xi_v)^{-\frac{1}{2}})^{\frac{2-2\alpha}{2-\alpha}}|\langle \varphi_\lambda, \mathbf{u}^{v,II}\rangle| \lesssim \lambda^{1-\alpha}\|\varphi_\lambda\|_{L^\infty}.$$

It remains to estimate the middle term,

$$t^{-\frac{1}{2}}|\chi_\lambda(v)\langle \varphi_\lambda, it(\phi'(\cdot/t)-\phi'(v))\mathbf{u}^v\rangle| \lesssim |\phi''(\lambda)|\cdot t^{\frac{1}{2}}a''(\lambda)^{\frac{1}{2}}\cdot\|\varphi_\lambda\|_{L^\infty} \lesssim t^{\frac{1}{2}}\lambda^{1-\frac{\alpha}{2}}\|\varphi_\lambda\|_{L^\infty}$$

This finishes the proof. □

We denote

$$\psi(t,x) = t^{-\frac{1}{2}}\chi_\lambda(x/t)\gamma(t,x/t)e^{it\phi(x/t)}.$$

We record some bounds for ψ:

Lemma 4.30. Assume that $(t,v)\in\mathscr{D}$. We have

$$\begin{aligned}
\|\partial_x^n\psi(t,x)\|_{L_x^\infty} &\lesssim \|\varphi_\lambda\|_{L_x^\infty}\lambda^n \\
\|\partial_x^n\psi(t,x)\|_{L_x^\infty} &\lesssim \lambda^{\frac{1}{2}-\frac{\alpha}{4}}t^{-1/4}\|\varphi_\lambda\|_{L_x^2}\lambda^n \\
\|\psi_x(t,x)\|_{L_x^r} &\lesssim \|\varphi_\lambda\|_{L_x^r}\lambda, \forall r\in[1,\infty]
\end{aligned}$$

Proof. We have

$$\psi_x(t,x) = \frac{1}{\sqrt{t}}\frac{1}{t}\chi_\lambda'\gamma(t,x/t)e^{it\phi(x/t)} + \frac{1}{\sqrt{t}}\chi_\lambda\frac{1}{t}\gamma_v(t,x/t)e^{it\phi(x/t)} + \frac{1}{\sqrt{t}}\chi_\lambda\gamma(t,x/t)e^{it\phi(x/t)}i\phi'(x/t)$$

By using Lemma 4.28, as well as the condition $(t,v)\in\mathscr{D}$, we have

$$\begin{aligned}
|\psi_x(t,x)| &\lesssim t^{-1/2}(t^{-1}t^{1/2}\|\varphi_\lambda\|_{L_x^\infty}(\lambda^{1-\alpha}+\lambda^{1-\frac{\alpha}{2}}t^{1/2}) + t^{1/2}\|\varphi_\lambda\|_{L_x^\infty}\lambda) \\
&\lesssim \|\varphi_\lambda\|_{L_x^\infty}(t^{-1}\lambda^{1-\alpha}+\lambda^{1-\alpha/2}t^{-1/2}+\lambda) \\
&\lesssim \lambda\|\varphi_\lambda\|_{L_x^\infty}
\end{aligned}$$

The other bounds can be deduced similarly. □

We also observe that on the wave packet scale, we may replace $\gamma(t,v)$ with $\gamma(t,x/t)$ up to acceptable errors. Indeed, let

$$\theta(t,x) = t^{-\frac{1}{2}}\chi_\lambda(x/t)\gamma(t,v)e^{it\phi(x/t)},$$

and denote

$$\beta_v^\lambda(t,x) = \theta(x,t) - \psi(t,x) = t^{-1/2}\chi_\lambda(x/t)(\gamma(t,v) - \gamma(t,x/t))e^{it\phi(x/t)},$$

We are also going to need some bounds for θ, that we record below:

Lemma 4.31. Assume that $(t,v) \in \mathscr{D}$. Then, we have the bounds

$$\|\theta(t,x)\|_{L_x^\infty} \lesssim \|\varphi_\lambda\|_{L_x^\infty}$$
$$\|\theta_x(t,x)\|_{L_x^r} \lesssim t^{1/r}\|\varphi_\lambda\|_{L_x^\infty}\lambda, \forall r \in [1,\infty]$$

Proof. We have

$$\theta_x(t,x) = \frac{1}{\sqrt{t}}\frac{1}{t}\chi_\lambda'(x/t)\gamma(t,v)e^{it\phi(x/t)} + \frac{1}{\sqrt{t}}\chi_\lambda(x/t)\gamma(t,v)e^{it\phi(x/t)}i\phi'(x/t)$$

By using Lemma 4.28, we have

$$|\theta_x(t,x)| \lesssim t^{-1/2}(t^{-1}\lambda^{1-\alpha}t^{1/2}\|\varphi_\lambda\|_{L_x^\infty} + t^{1/2}\|\varphi_\lambda\|_{L_x^\infty}\lambda) \lesssim \|\varphi_\lambda\|_{L_x^\infty}\lambda$$

The other bounds can be deduced similarly. □

Lemma 4.32. Let $v \in J_\lambda$, and $(t,v) \in \mathscr{D}$. Then, for every $y \neq 0$ and x such that $|x - vt| \lesssim \delta x = t^{1/2}\lambda^{\frac{\alpha}{2}-1}$, we have the bound

$$|\delta^y\beta_v| \lesssim t^{-3/4}\lambda^{\frac{\alpha}{4}-\frac{1}{2}}\|\varphi\|_X$$

Proof. We have

$$\delta^y\beta_v = -t^{-1/2}\delta^y(\gamma(t,\cdot/t))\chi_\lambda((x+y)/t)e^{it\phi((x+y)/t)} + t^{-1/2}(\gamma(t,v) - \gamma(t,x/t))\delta^y(\chi_\lambda(\cdot/t)e^{it\phi(\cdot/t)}).$$

The Mean Value Theorem, together with the third bound from Proposition 4.28 and the condition $(t,v) \in \mathscr{D}$ ensure that

$$|\delta^y(\gamma(t,\cdot/t))| \lesssim t^{-1}\|\partial_v\gamma\|_{L^\infty},$$

and that

$$\begin{aligned}
|\delta^y\beta_v| &\lesssim t^{-1/2}(t^{-1}\|\partial_v\gamma\|_{L^\infty} + t^{-1/2}\lambda^{\frac{\alpha}{2}-1}\|\partial_v\gamma\|_{L^\infty}(t^{-1}\lambda^{1-\alpha} + \lambda)) \\
&\lesssim t^{-1}\|\partial_v\gamma\|_{L^\infty}(t^{-1/2} + \lambda^{\alpha/2}) \lesssim t^{-1}\lambda^{\alpha/2}(t^{\frac{1}{4}}\lambda^{-\frac{1}{2}-\frac{\alpha}{4}}\|\varphi\|_X + t^{\frac{1}{2}}\|\varphi_\lambda\|_{L_x^\infty}) \\
&\lesssim t^{-3/4}\lambda^{\frac{\alpha}{4}-\frac{1}{2}}\|\varphi\|_X + t^{-1/2}\lambda^{\alpha/2}\|\varphi_\lambda\|_{L_x^\infty} \lesssim t^{-3/4}\lambda^{\frac{\alpha}{4}-\frac{1}{2}}\|\varphi\|_X
\end{aligned}$$

□

4.32.1 Bounds for Q

Write, slightly abusing notation,

$$Q(\varphi) = Q(\varphi, \overline{\varphi}, \varphi) := \frac{1}{3} \int \frac{1}{|y|^{\alpha-1}} \operatorname{sgn}(y) \cdot |\delta^y \varphi|^2 \delta^y \varphi \, dy.$$

Lemma 4.33. For $\alpha > 1$, we have the difference estimates

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_x} \lesssim \|\partial_x(\varphi_1, \varphi_2)\|_{L^\infty_x} \|(\varphi_1, \varphi_2)\|_{W^{1,\infty}_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^2_x} \lesssim \|\partial_x(\varphi_1, \varphi_2)\|_{L^2_x} \|(\varphi_1, \varphi_2)\|_{W^{1,\infty}_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

while for $\alpha < 1$ we have

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_x} \lesssim \|(\varphi_1, \varphi_2)\|_{L^\infty_x} \|(\varphi_1, \varphi_2)\|_{W^{1,\infty}_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^2_x} \lesssim \|(\varphi_1, \varphi_2)\|_{L^2_x} \|(\varphi_1, \varphi_2)\|_{W^{1,\infty}_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

Moreover, for $\alpha > 1$ we also have the estimates

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_x} \lesssim \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{W^{2\delta,\infty}_x} \||D_x|^{\alpha-1}(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^2_x} \lesssim \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{H^{2\delta}_x} \||D_x|^{\alpha-1}(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

while for $\alpha < 1$ we have

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^\infty_x} \lesssim \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{W^{\delta,\infty}_x} \||D_x|^{\alpha-\delta}(\varphi_1, \varphi_2)\|_{W^{\delta,\infty}_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

$$\|Q(\varphi_1) - Q(\varphi_2)\|_{L^2_x} \lesssim \||D_x|^{1-\delta}(\varphi_1, \varphi_2)\|_{H^{\delta}_x} \||D_x|^{\alpha-\delta}(\varphi_1, \varphi_2)\|_{H^{\delta}_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x},$$

Proof. We only prove the first two estimates in the case $\alpha > 1$, as the other ones are similar.

Write

$$Q(\varphi_1) - Q(\varphi_2) = \int_{|y|\leq 1} + \int_{|y|>1}$$

where the integrand may be written

$$\begin{aligned} \operatorname{sgn}(y)(|\delta^y\varphi_1|^2\delta^y\overline{\varphi_1} - &|\delta^y\varphi_2|^2\delta^y\overline{\varphi_2}) \\ &= \operatorname{sgn}(y)(\delta^y(\varphi_1 - \varphi_2)(|\delta^y\varphi_1|^2 + |\delta^y\varphi_2|^2) + \delta^y(\overline{\varphi_1} - \overline{\varphi_2})\delta^y\varphi_1\delta^y\varphi_2). \end{aligned}$$

The first integral contributes to the two estimates respectively,

$$\left| \int_{|y|\leq 1} \right| \lesssim \|\partial_x(\varphi_1, \varphi_2)\|^2_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}$$

and

$$\left\| \int_{|y|\leq 1} \right\|_{L^2} \lesssim \|\partial_x(\varphi_1, \varphi_2)\|_{L^2_x} \|\partial_x(\varphi_1, \varphi_2)\|_{L^\infty_x} \|\partial_x(\varphi_1 - \varphi_2)\|_{L^\infty_x}.$$

For the second,

$$\left\| \int_{|y|>1} \right\|_{L_x^\infty} \lesssim \|\|\partial_x(\varphi_1,\varphi_2)\|_{L_x^\infty}\|(\varphi_1,\varphi_2)\|_{L_x^\infty}\|\partial_x(\varphi_1-\varphi_2)\|_{L_x^\infty}$$

and

$$\left\| \int_{|y|>1} \right\|_{L^2} \lesssim \|\|\partial_x(\varphi_1,\varphi_2)\|_{L_x^2}\|(\varphi_1,\varphi_2)\|_{L_x^\infty}\|\partial_x(\varphi_1-\varphi_2)\|_{L_x^\infty}.$$

□

We will be considering separately the balanced and unbalanced components of Q. Precisely, we denote the diagonal set of frequencies by $\mathscr{D}$ and write

$$\begin{aligned} Q(\varphi,\varphi,\varphi) &= \sum_{(\lambda_1,\lambda_2,\lambda_3,\lambda)\in\mathscr{D}} Q(\varphi_{\lambda_1},\varphi_{\lambda_2},\varphi_{\lambda_3}) + \sum_{(\lambda_1,\lambda_2,\lambda_3,\lambda)\notin\mathscr{D}} Q(\varphi_{\lambda_1},\varphi_{\lambda_2},\varphi_{\lambda_3}) \\ &= Q^{bal}(\varphi,\varphi,\varphi) + Q^{unbal}(\varphi,\varphi,\varphi) = Q^{bal}(\varphi) + Q^{unbal}(\varphi). \end{aligned}$$

Here, λ is the frequency of the output of $Q(\varphi_{\lambda_1},\varphi_{\lambda_2},\varphi_{\lambda_3})$, and the diagonal set $\mathscr{D}$ is defined by $\mathscr{D} = \{(\lambda_1,\lambda_2,\lambda_3,\lambda)|\lambda_1\sim\lambda_2\sim\lambda_3\sim\lambda\}$. The unbalanced portion of Q satisfies the better bound as follows:

Lemma 4.34. Q^{unbal} satisfies the bounds

$$\|\chi_\lambda^1\partial_x P_\lambda Q^{unbal}(\varphi)\|_{L_x^\infty} \lesssim \lambda^{\max\{\frac{1-\alpha}{2},0\}-\delta}\frac{\|\varphi\|_X^3}{t^2}$$

and

$$\|\chi_\lambda^1\partial_x P_\lambda Q^{unbal}(\varphi)\|_{L_x^2} \lesssim \lambda^{-\delta}\frac{\|\varphi\|_X^3}{t^{3/2}},$$

where χ_λ^1 is a cut-off widening χ_λ.

Proof. We shall denote

$$I_{\lambda_1,\lambda_2,\lambda_3} = \int_{\mathbb{R}} \frac{1}{|y|^{\alpha-1}}\operatorname{sgn}(y)\delta^y\varphi_{\lambda_1}\delta^y\varphi_{\lambda_2}\delta^y\varphi_{\lambda_3}\,dy$$

and consider two cases in the frequency sum for $\partial_x P_\lambda Q^{unbal}$.

First we consider the case in which we have two low separated frequencies. We assume without loss of generality that $\lambda_3=\lambda$ and $\lambda_1<\lambda_2\ll\lambda$. We analyze the case $\alpha>1$. Here, the elliptic estimates will be applied for the factor φ_{λ_2}. Precisely, from Lemma 4.6 and estimates 4.25.6, 4.25.7, and 4.25.8, we get that

$$\begin{aligned} \left\|\chi_\lambda^1 I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} &\lesssim \frac{\lambda_2^{\frac{3}{2}-\alpha}}{t}\|\varphi\|_X(\lambda_1^{1-2\delta}+\lambda_1)\|\varphi_{\lambda_1}\|_{L_x^\infty}\lambda_3^{\alpha+\delta-1}\|\varphi_{\lambda_3}\|_{L_x^\infty} \\ &\lesssim \frac{\lambda_2^{\frac{3}{2}-\alpha}}{t}\|\varphi\|_X\lambda_1^{\frac{\alpha-1}{2}+\delta}(\lambda_1^{\frac{3-\alpha}{2}-3\delta}+\lambda_1^{\frac{3-\alpha}{2}-\delta})\|\varphi_{\lambda_1}\|_{L_x^\infty}\lambda^{-\left(2-\frac{\alpha}{2}+2\delta\right)}\lambda^{1+\frac{\alpha}{2}+3\delta}\|\varphi_\lambda\|_{L_x^\infty} \\ &\lesssim \lambda_2^{\frac{3}{2}-\alpha}\lambda_1^{\frac{\alpha-1}{2}+\delta}\lambda^{-\left(2-\frac{\alpha}{2}+2\delta\right)}\frac{\|\varphi\|_X^3}{t^2}. \end{aligned}$$

When $\alpha < 1$, we also apply the elliptic estimates for the factor φ_{λ_1}. Precisely, from Lemma 4.6 and estimates 4.25.6, 4.25.7, and 4.25.8, we get that

$$\begin{aligned}\left\|\chi_\lambda^1 I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} &\lesssim \frac{\lambda_1^{\frac{3}{2}-\alpha}}{t}\|\varphi\|_X(\lambda_2^{2\delta}+\lambda_2^{4\delta})\|\varphi_{\lambda_2}\|_{L_x^\infty}\lambda_3^{\alpha-3\delta}\|\varphi_{\lambda_3}\|_{L_x^\infty}\\ &\lesssim \frac{\lambda_1^{\frac{3}{2}-\alpha}}{t}\|\varphi\|_X\lambda_2^{-(1+\delta)}(\lambda_2^{1+\frac{\alpha}{2}+3\delta}+\lambda_2^{1+\frac{\alpha}{2}+5\delta})\|\varphi_{\lambda_2}\|_{L_x^\infty}\lambda^{-\left(1-\frac{\alpha}{2}+4\delta\right)}\lambda^{1+\frac{\alpha}{2}+\delta}\|\varphi_\lambda\|_{L_x^\infty}\\ &\lesssim \lambda_1^{\frac{3}{2}-\alpha}\lambda_2^{-(1+\delta)}\lambda^{-\left(1-\frac{\alpha}{2}+2\delta\right)}\frac{\|\varphi\|_X^3}{t^2}.\end{aligned}$$

By using dyadic summation in λ_1 and λ_2, we deduce that

$$\left\|\chi_\lambda^1\partial_x\sum_{\lambda_1<\lambda_2\ll\lambda}I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty}\lesssim \lambda^{\max\{\frac{1-\alpha}{2},0\}-\delta}\frac{\|\varphi\|_X^3}{t^2}.$$

Similarly, we deduce that

$$\left\|\chi_\lambda^1\partial_x\sum_{\lambda_1<\lambda_2\ll\lambda}I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^2}\lesssim \lambda^{-\delta}\frac{\|\varphi\|_X^3}{t^{3/2}}$$

We now analyze the situation in which $\lambda_1,\lambda_2\gtrsim\lambda$, and λ_1 and λ_2 are comparable and both separated from λ. Thus, we will be able to use λ_1 and λ_2 interchangeably. We replace χ_λ^1 by $\tilde{\chi}_\lambda$, which has double support, and equals 1 on a comparably-sized neighbourhood of the support of χ_λ^1. We write

$$\chi_\lambda^1\partial_x P_\lambda = \chi_\lambda^1\partial_x P_\lambda\tilde{\chi}_\lambda + \chi_\lambda^1\partial_x P_\lambda(1-\tilde{\chi}_\lambda).$$

For the first term, when $\alpha > 1$ using Lemma 4.6, along with estimates 4.25.6, 4.25.7, 4.25.8, we get the bounds

$$\begin{aligned}\left\|\chi_\lambda^1 P_\lambda\tilde{\chi}_\lambda I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} &\lesssim \lambda_2^{1/2+\delta}\frac{\lambda_3^{1-2\delta}+\lambda_3}{t}\|\varphi\|_X\|\varphi_{\lambda_2}\|_{L_x^\infty}\|\varphi_{\lambda_3}\|_{L_x^\infty}\\ &\lesssim \lambda_2^{-1-3\delta/2}\lambda_3^{\delta/2}\frac{\|\varphi\|_X}{t}(\lambda_3^{1-5\delta/2}+\lambda_3^{1-\delta/2})\|\varphi_{\lambda_3}\|_{L_x^\infty}\lambda_2^{\frac{3}{2}+5\delta/2}\|\varphi_{\lambda_2}\|_{L_x^\infty}\\ &\lesssim \lambda_2^{-1-\frac{3\delta}{2}}\lambda_3^{\delta/2}\frac{\|\varphi\|_X^3}{t^2},\end{aligned}$$

and

$$\begin{aligned}\left\|\chi_\lambda^1 P_\lambda\tilde{\chi}_\lambda I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} &\lesssim \lambda_2^{1/2+\delta}\frac{\lambda_3^{1-2\delta}+\lambda_3}{t}\|\varphi\|_X\|\varphi_{\lambda_2}\|_{L_x^\infty}\|\varphi_{\lambda_3}\|_{L_x^\infty}\\ &\lesssim \lambda_2^{-\frac{\alpha+1}{2}-3\delta/2}\lambda_3^{\delta/2}\frac{\|\varphi\|_X}{t}(\lambda_3^{1-5\delta/2}+\lambda_3^{1-\delta/2})\|\varphi_{\lambda_3}\|_{L_x^\infty}\lambda_2^{\frac{3}{2}+5\delta/2}\|\varphi_{\lambda_2}\|_{L_x^\infty}\\ &\lesssim \lambda_2^{-\frac{\alpha+1}{2}-\frac{3\delta}{2}}\lambda_3^{\delta/2}\frac{\|\varphi\|_X^3}{t^2},\end{aligned}$$

when $\alpha < 1$.

We similarly get the following L_x^2 bound:

$$\begin{aligned}
\left\|\chi_\lambda^1 P_\lambda \tilde{\chi}_\lambda I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^2} &\lesssim \lambda_2^\delta \frac{\lambda_3^{1-2\delta}+\lambda_3}{t}\|\varphi\|_X \|\varphi_{\lambda_2}\|_{L_x^\infty}\|\varphi_{\lambda_3}\|_{L_x^\infty} \\
&\lesssim \lambda_2^{-1-3\delta/2}\lambda_3^{\delta/2}\frac{\|\varphi\|_X}{t}(\lambda_3^{1-5\delta/2}+\lambda_3^{1-\delta/2})\|\varphi_{\lambda_3}\|_{L_x^\infty}\lambda_2^{1+5\delta/2}\|\varphi_{\lambda_2}\|_{L_x^\infty} \\
&\lesssim \lambda_2^{-1-3\delta/2}\lambda_3^{\delta/2}\frac{\|\varphi\|_X^3}{t^2}.
\end{aligned}$$

By using dyadic summation in λ_1, λ_2, and λ_3 (and by using the fact that λ_1 and λ_2 are close), we deduce the bounds

$$\left\|\chi_\lambda^1 \partial_x P_\lambda \tilde{\chi}_\lambda \sum_{\lambda_3 \lesssim \lambda_2, \lambda_1 \simeq \lambda_2 \gtrsim \lambda} I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^\infty} \lesssim \lambda^{\max\{\frac{1-\alpha}{2},0\}-\delta}\frac{1}{t^2}\|\varphi\|_X^3$$

$$\left\|\chi_\lambda^1 \partial_x P_\lambda \tilde{\chi}_\lambda \sum_{\lambda_3 \lesssim \lambda_2, \lambda_1 \simeq \lambda_2 \gtrsim \lambda} I_{\lambda_1,\lambda_2,\lambda_3}\right\|_{L_x^2} \lesssim \lambda^{-\delta}\frac{1}{t^2}\|\varphi\|_X^3.$$

We look at the second term. For every N, we know that

$$\|\chi_\lambda^1 \partial_x P_\lambda (1-\tilde{\chi}_\lambda)\|_{L^2 \to L^2}, \|\chi_\lambda^1 \partial_x P_\lambda (1-\tilde{\chi}_\lambda)\|_{L^\infty \to L^\infty} \lesssim \frac{\lambda^{1-N}}{t^N}$$

By carrying out a similar analysis as above, along with Lemma 4.6 and dyadic summation, we deduce that the contributions corresponding to these terms are also acceptable. □

Lemma 4.35. We have

$$\chi_\lambda((x/t))^3 Q(e^{it\phi(x/t)}) = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} q(\phi'(x/t)) + h(\lambda,t),$$

where for every $a \in (0,1)$

$$|h(\lambda,t)| \lesssim \frac{\lambda^{6-3\alpha}+\lambda^{5-2\alpha}}{t^{2-(4-\alpha)a}} + \frac{\lambda^{3-\alpha}}{t^{1-(2-\alpha)a}} + \frac{1}{t^{(1+\alpha)a}}$$

when $\alpha > 1$, and

$$|h(\lambda,t)| \lesssim \frac{\lambda^{4-2\alpha}}{t^{2-(3-\alpha)a}} + \frac{\lambda^{3-\alpha}}{t^{1-(2-\alpha)a}} + \frac{1}{t^{(1+\alpha)a}}$$

for $\alpha < 1$.

Proof. We write

$$e^{-it\phi(x/t)}\delta^y e^{it\phi(x/t)} = \frac{e^{iy\phi'(x/t)}\left(e^{it/2\phi''(c_{x,y}/t)y^2/t^2}-1\right)}{y} + \frac{e^{iy\phi'(x/t)}-1}{y} =: a+b,$$

where $c_{x,y}$ is between x and $x+y$. We now use the fact that x/t belongs to the support of χ_λ. We have

$$|\chi_\lambda(x/t)||b| \lesssim \lambda.$$

Moreover, when $|y|\leq t^a$, $|c_{x,y}/t - x/t| \leq |y/t| \leq t^{a-1}$. This implies that $c_{x,y}/t$ belongs to the support of the enlarged cut-off χ^1_λ, hence $\phi''(c_{x,y}/t) \simeq \lambda^{2-\alpha}$. We note the bound

$$|\chi_\lambda(x/t)||a| \lesssim |\chi_\lambda(x/t)||y/(2t)\phi''(c_{x,y}/t)| \left|\frac{e^{\pm i\phi''(c_{x,y}/t)y^2/(2t)}-1}{\phi''(c_{x,y}/t)y^2/(2t)}\right| \lesssim \lambda^{2-\alpha}t^{a-1}$$

Here, we have used the fact that the function $x \to \dfrac{e^{ix}-1}{x}$, where $x\in\mathbb{R}$, is bounded.

Thus, we have the bounds

$$\begin{aligned} |\chi_\lambda(x/t)||a| &\lesssim \lambda^{2-\alpha}t^{a-1} \\ |\chi_\lambda(x/t)||b| &\lesssim \lambda \end{aligned} \tag{4.35.1}$$

We also note the cruder bounds

$$|\chi_\lambda(x/t)||a| + |\chi_\lambda(x/t)||b| \lesssim \frac{1}{|y|} \tag{4.35.2}$$

We write

$$\begin{aligned} (\chi_\lambda(x/t))^3 Q(e^{it\phi(x/t)}) &= (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int \frac{1}{|y|^{\alpha-1}}\,|b|^2\,b\,dy \\ &\quad + (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int \frac{1}{|y|^{\alpha-1}}(a^2\overline{a} + a^2\overline{b} + 2|a|^2 b + 2a|b|^2 + b^2\overline{a})\,dy \\ &:= T_1 + T_2 \end{aligned}$$

We note that

$$T_1 = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} \int \frac{1}{|y|^{\alpha-1}}\,|b|^2\,b\,dy = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} q(\phi'(x/t)),$$

so we only need to analyze T_2.

We first bound the contribution over the region $|y|\leq t^a$, which we shall denote by T_2^1. We denote the contribution over the region $|y|>t^a$ by T_2^2. We have

$$\begin{aligned} T_2^1 = (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} &\int_{|y|\leq t^a} \frac{1}{|y|^{\alpha-1}}(a^2\overline{a} + a^2\overline{b} + 2|a|^2 b)\,dy \\ + (\chi_\lambda(x/t))^3 e^{it\phi(x/t)} &\int_{|y|\leq t^a} \frac{1}{|y|^{\alpha-1}}(2a|b|^2 + b^2\overline{a})\,dy := T_{2a} + T_{2b}, \end{aligned}$$

4.35.1 implies that

$$|T_{2a}| \lesssim |\chi_\lambda(x/t)|^3 \int_{|y|\le t^a} \frac{1}{|y|^{\alpha-1}}(|a|^3 + |a|^2|b|)\, dy \lesssim \int_{|y|\le t^a} \frac{1}{|y|^{\alpha-1}} \frac{\lambda^{4-2\alpha}}{t^{2-2a}} \left(\frac{\lambda^{2-\alpha}}{t^{1-a}} + \lambda\right)$$
$$\lesssim \int_{|y|\le t^a} \frac{1}{|y|^{\alpha-1}} \frac{\lambda^{6-3\alpha}+\lambda^{5-2\alpha}}{t^{2-2a}}\, dy \lesssim t^{a(2-\alpha)} \frac{\lambda^{6-3\alpha}+\lambda^{5-2\alpha}}{t^{2-2a}} \lesssim \frac{\lambda^{6-3\alpha}+\lambda^{5-2\alpha}}{t^{2-(4-\alpha)a}},$$

when $\alpha > 1$, and

$$|T_{2a}| \lesssim |\chi_\lambda(x/t)|^3 \int_{|y|\le t^a} \frac{1}{|y|^{\alpha-1}}(|a|^3 + |a|^2|b|)\, dy \lesssim \int_{|y|\le t^a} \frac{1}{|y|^{\alpha}} \frac{\lambda^{2-\alpha}}{t^{1-a}} \left(\frac{\lambda^{2-\alpha}}{t^{1-a}} + \lambda\right)$$
$$\lesssim \int_{|y|\le t^a} \frac{1}{|y|^{\alpha}} \left(\frac{\lambda^{4-2\alpha}}{t^{2-2a}} + \frac{\lambda^{3-\alpha}}{t^{1-a}}\right) dy \lesssim t^{a(1-\alpha)} \left(\frac{\lambda^{4-2\alpha}}{t^{2-2a}} + \frac{\lambda^{3-\alpha}}{t^{1-a}}\right) \lesssim \frac{\lambda^{4-2\alpha}}{t^{2-(3-\alpha)a}} + \frac{\lambda^{3-\alpha}}{t^{1-(2-\alpha)a}},$$

when $\alpha < 1$ (here, we have used the cruder bounds (4.35.2) for the first factor, and (4.35.1) for the rest.

4.35.1 and 4.35.2 imply the bound

$$|\chi_\lambda(x/t)|^3|b|^2|a| = |\chi_\lambda(x/t)|^3\, |b|^2\, |y/(2t)\phi''(c_{x,y}/t)| \left|\frac{e^{\pm i\phi''(c_{x,y}/t)y^2/(2t)} - 1}{\phi''(c_{x,y}/t)y^2/(2t)}\right|$$
$$\lesssim \lambda \frac{1}{|y|} |\chi_\lambda(x/t)||y/(2t)\phi''(c_{x,y}/t)| \lesssim \frac{\lambda^{3-\alpha}}{t}$$

It follows that T_{2b} satisfies the bound

$$|T_{2b}| \lesssim |\chi_\lambda(x/t)|^3 \int_{|y|\le t^a} \frac{1}{|y|^{\alpha-1}}\, |b|^2\, |a|\, dy \lesssim \int_{|y|\le t^a} \frac{1}{|y|^{\alpha-1}} \frac{\lambda^{3-\alpha}}{t}\, dy \lesssim \frac{\lambda^{3-\alpha}}{t^{1-(2-\alpha)a}}$$

For T_2^2, 4.35.2 implies that

$$|T_2^2| \lesssim \int_{|y|>t^a} \frac{1}{|y|^{\alpha+2}}\, dy \lesssim \frac{1}{t^{(1+\alpha)a}}.$$

This finishes the proof. □

4.35.1 The asymptotic equation for γ

Here we prove the following:

Proposition 4.36. Let $v \in J_\lambda$. Under the assumption $(t, v) \in \mathscr{D}$, we have

$$\dot\gamma(t, v) = iq(\xi_v)\xi_v t^{-1}\gamma(t, v)|\gamma(t, v)|^2 + f(t, v),$$

where

$$|f(t,v)| \lesssim \lambda^{-\delta} g(\lambda,\alpha) t^{-1-\delta+C\epsilon^2}\epsilon,$$

where $g = g(\lambda,\alpha)$ is a sum of powers of λ that might also depend on α.

Moreover, we also have the following L^2_v bound:

$$\|f(t,v)\|_{L^2_v(J_\lambda)} \lesssim (\lambda^{-\delta} + \lambda^{-\frac{3}{2}-\delta}) t^{-1-\delta+C\epsilon^2}\epsilon.$$

Proof. We have

$$\dot\gamma(t,v) = \langle \dot\varphi, \mathbf{u}^v_\lambda\rangle + \left\langle \varphi, \dot{\mathbf{u}^v_\lambda}\right\rangle = \langle P_\lambda A_\varphi \varphi, \mathbf{u}^v\rangle + i\langle \varphi_\lambda, (i\partial_t - A(D))\mathbf{u}^v\rangle := I_1 + I_2.$$

We first analyze I_2. We use Lemma 4.27 to write

$$(i\partial_t - A(D))\mathbf{u}^v = t^{-\frac{3}{2}}(L\mathbf{u}^{v,I} + \mathbf{r}^v)$$

$$\begin{aligned}
|\langle \varphi_\lambda, (i\partial_t - A(D))\mathbf{u}^v\rangle| &\lesssim t^{-\frac{3}{2}}(\|L\varphi_\lambda\|_{L^2_x}\cdot \lambda^{1-\frac{\alpha}{2}}\lambda^{\frac{1}{2}-\frac{\alpha}{4}}t^{1/4} + \|\varphi_\lambda\|_{L^2_x}\cdot\lambda^{-\alpha/2}\lambda^{\frac{1}{2}-\frac{\alpha}{4}}t^{1/4})\\
&\lesssim \lambda^{\frac{1}{2}-\frac{3\alpha}{4}} t^{-5/4}(1+\lambda^{-1})\|\varphi\|_X\\
&\lesssim \lambda^{\frac{1-\alpha}{2}-\delta} t^{-1-\delta}(1+\lambda^{-1})\epsilon t^{C\epsilon^2}
\end{aligned}$$

and

$$\begin{aligned}
\|\chi_\lambda\langle \varphi_\lambda, (i\partial_t - A(D))\mathbf{u}^v\rangle\|_{L^2_v} &\lesssim t^{-3/2}\left(\|L\varphi_\lambda\|_{L^2_x}\lambda^{1-\frac{\alpha}{2}} + \|\varphi_\lambda\|_{L^2_x}\lambda^{-\frac{\alpha}{2}}\right)\\
&\lesssim \lambda^{-\alpha/4}t^{-5/4}t^{-1/4}\lambda^{-\alpha/4}(1+\lambda^{-1})\|\varphi\|_X\\
&\lesssim \lambda^{-\delta}t^{-1-\delta}(1+\lambda^{-1})\epsilon t^{C\epsilon^2}
\end{aligned}$$

(we have used the condition $(t,v)\in\mathscr{D}$.)

In the remaining part of this section we shall analyze the term I_1. We first exchange F for its principal quadratic term, expanding

$$F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2 = \int_0^1 \frac{(1-h)^2}{2}(\delta^y\varphi)^3 F'''(h\delta^y\varphi)\,dh.$$

When $\alpha<1$, from Bernstein's inequality, Moser's estimate (the nonlinear version, as well as the one for products), Lemma 4.6, and Sobolev embedding and interpolation, we get that

$$\begin{aligned}
&\lambda^{\frac{\alpha}{2}-\frac{1}{2}}\left\|P_\lambda\int \frac{1}{|y|^{\alpha-1}}(\delta^y\varphi)^3F'''(h\delta^y\varphi)|\delta|^y\varphi_x\,dy\right\|_{L^\infty_x}\\
&\lesssim \int \frac{1}{|y|^{\alpha-1}}\||D_x|^{\frac{\alpha}{2}}((\delta^y\varphi)^3|\delta|^y\varphi_xF'''(h\delta^y\varphi))\|_{L^2_x}\,dy\\
&\lesssim \|\varphi_x\|_{L^\infty_x}\||D_x|^{\frac{\alpha}{2}}\varphi_x\|_{L^2_x}(\|\varphi_x\|^3_{L^\infty_x} + \|\varphi_x\|^2_{L^\infty_x}\||D_x|^\delta\varphi\|_{L^\infty_x})\\
&\lesssim \frac{1}{t^2}\epsilon^5\langle t\rangle^{C\epsilon^2}.
\end{aligned}$$

When $\alpha > 1$, we have

$$\begin{aligned}
&\lambda^{\delta}\left\|P_\lambda \int \frac{1}{|y|^{\alpha-1}}(\delta^y\varphi)^3 F'''(h\delta^y\varphi)|\delta|^y\varphi_x\, dy\right\|_{L^\infty_x} \\
&\lesssim \int \frac{1}{|y|^{\alpha-1}}\||D_x|^{2\delta}((\delta^y\varphi)^3|\delta|^y\varphi_x F'''(h\delta^y\varphi))\|_{L_x^{1/\delta}}\, dy \\
&\lesssim \|\varphi_x\|^4_{L^\infty_x}(\||D_x|^{\frac{1}{2}+\delta}\varphi_x\|_{L^2_x} + \||D_x|^{\frac{1}{2}+\delta}\varphi_{xx}\|_{L^2_x}) \\
&+ \|\varphi_x\|^3_{L^\infty_x}\||D_x|^{\frac{1}{2}+\delta}\varphi_x\|_{L^2_x}(\|\varphi_x\|_{L^\infty_x} + \||D_x|^{\alpha-1+\delta}\varphi_x\|_{L^\infty_x}) \\
&\lesssim \frac{1}{t^2}\epsilon^5\langle t\rangle^{C\epsilon^2}.
\end{aligned}$$

We have also used Sobolev embedding and the classical Moser estimate together with Bernstein's inequality, keeping in mind $F'''(0) = 0$. Similarly,

$$\left\|\int_{\mathbb{R}} \frac{1}{|y|^{\alpha-1}}P_\lambda((F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2)|\delta|^y\varphi_x)\, dy\right\|_{L^2_x} \lesssim \lambda^{-\delta}\frac{1}{t^{3/2}}\epsilon^5\langle t\rangle^{C\epsilon^2}.$$

By Hölder's inequality and Young's inequality respectively,

$$\left|\left\langle P_\lambda \int_{\mathbb{R}} \frac{1}{|y|^{\alpha-1}}\left(F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2\right)|\delta|^y\varphi_x\, dy, \varphi_v\right\rangle\right| \lesssim \lambda^{\max\{\frac{1-\alpha}{2},0\}-\delta}\frac{1}{t^{3/2}}\epsilon^5\langle t\rangle^{C\epsilon^2},$$

$$\left\|\left\langle P_\lambda \int_{\mathbb{R}} \frac{1}{|y|^{\alpha-1}}\left(F(\delta^y\varphi) - \frac{1}{2}(\delta^y\varphi)^2\right)|\delta|^y\varphi_x\, dy, \varphi_v\right\rangle\right\|_{L^2_v(J_\lambda)} \lesssim \lambda^{-\delta}\frac{1}{t^{3/2}}\epsilon^5\langle t\rangle^{C\epsilon^2}.$$

We are left to estimate

$$\begin{aligned}
\left\langle P_\lambda \int_{\mathbb{R}} \frac{1}{|y|^{\alpha-1}}(\delta^y\varphi)^2|\delta|^y\varphi_x\, dy, \mathbf{u}^v\right\rangle &= \left\langle \partial_x P_\lambda Q^{\mathrm{bal}}(\varphi), \mathbf{u}^v\right\rangle + \left\langle \chi^1_\lambda\partial_x P_\lambda Q^{\mathrm{unbal}}(\varphi), \mathbf{u}^v\right\rangle \\
&+ \left\langle (1-\chi^1_\lambda)\partial_x P_\lambda Q^{\mathrm{unbal}}(\varphi), \mathbf{u}^v\right\rangle,
\end{aligned}$$

where χ^1_λ be a cut-off function enlarging χ_λ. Due to the fact that $\mathbf{u}^v$ is supported in the region $\left|\frac{x}{t} - v\right| \lesssim \lambda^{\frac{\alpha}{2}-1}t^{-1/2}$, the condition $(t,v) \in \mathscr{D}$ will imply that the third term is identically zero, while Lemma 4.34 implies that the second term is an acceptable error. Thus, we only have to analyze

$$\left\langle \partial_x P_\lambda Q^{\mathrm{bal}}(\varphi), \mathbf{u}^v\right\rangle = \langle \partial_x P_\lambda Q(\varphi_\lambda), \mathbf{u}^v\rangle.$$

Let χ^1 be a cut-off function that is equal to 1 on the support of the wave packet $\mathbf{u}_v$. Let $\tilde{\chi}$ be another cut-off function whose support is slightly larger than the one of χ^1, but comparable in size, uniformly in λ. We write

$$\langle \partial_x P_\lambda Q(\varphi_\lambda), \mathbf{u}^v\rangle = \langle \partial_x P_\lambda\tilde{\chi}Q(\varphi_\lambda), \mathbf{u}^v\rangle + \left\langle \chi^1\partial_x P_\lambda(1-\tilde{\chi})Q(\varphi_\lambda), \mathbf{u}^v\right\rangle$$

As in the proof of Lemma 4.34, we note that the operator norm bounds

$$\|\chi^1\partial_x P_\lambda(1-\tilde{\chi})\|_{L^\infty\to L^\infty} + \|\chi^1\partial_x P_\lambda(1-\tilde{\chi})\|_{L^2\to L^2} \lesssim \lambda^{1-2N}t^{-N}$$

for every N imply that the second term is acceptable error. This leaves us with the first.

We first replace φ_λ by $\chi_\lambda\varphi_\lambda$. When $\alpha > 1$, from Lemma 4.33, we have

$$\begin{aligned}
&|\langle \partial_x P_\lambda \tilde{\chi}(Q(\varphi_\lambda) - Q(\chi_\lambda\varphi_\lambda)), \mathbf{u}^v\rangle| \\
&\quad \lesssim \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\varphi_\lambda)\|_{L^\infty_x}\|(|D_x|^{\alpha-1}(\chi_\lambda\varphi_\lambda), |D_x|^{\alpha-1}\varphi_\lambda)\|_{L^\infty_x}\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x} \\
&\quad + \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\varphi_\lambda)\|_{L^\infty_x}\|(\chi_\lambda\varphi_\lambda, \varphi_\lambda)\|_{L^\infty_x}\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x},
\end{aligned}$$

while for $\alpha < 1$, the same lemma implies that

$$\begin{aligned}
&\lambda^{\frac{\alpha-1}{2}}|\langle \partial_x P_\lambda \tilde{\chi}(Q(\varphi_\lambda) - Q(\chi_\lambda\varphi_\lambda)), \mathbf{u}^v\rangle| \\
&\qquad \lesssim \lambda^{\frac{\alpha+1}{2}}\|((\chi_\lambda\varphi_\lambda), \varphi_\lambda)\|^2_{L^\infty_x}\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x} \\
&\qquad + \lambda^{\frac{\alpha+1}{2}}\|(|D_x|^{\alpha+\delta}(\chi_\lambda\varphi_\lambda), |D_x|^{\alpha+\delta}\varphi_\lambda)\|_{L^\infty_x}\|(\chi_\lambda\varphi_\lambda, \varphi_\lambda)\|_{L^\infty_x}\|\partial_x((1-\chi_\lambda)\varphi_\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x}.
\end{aligned}$$

By interpolation, Moser's estimates for the fractional derivatives, along with Lemma 4.26 and the condition $(t,v) \in \mathscr{D}$, it follows that the errors are acceptable. The L^2_x-bound is similar.

We now replace $\chi_\lambda\varphi_\lambda$ by ψ. From Lemma 4.33, when $\alpha > 1$, we have

$$\begin{aligned}
&|\langle \partial_x P_\lambda \tilde{\chi}(Q(\chi_\lambda\varphi_\lambda) - Q(\psi)), \mathbf{u}^v\rangle| \\
&\qquad\qquad \lesssim \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\psi)\|^2_{L^\infty_x}\|\partial_x(\chi_\lambda(x/t)r^\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x} \\
&\qquad\qquad + \lambda\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\psi)\|_{L^\infty_x}\|(\chi_\lambda\varphi_\lambda, \psi)\|_{L^\infty_x}\|\partial_x(\chi_\lambda(x/t)r^\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda^{\frac{\alpha-1}{2}}|\langle \partial_x P_\lambda \tilde{\chi}(Q(\chi_\lambda\varphi_\lambda) - Q(\psi)), \mathbf{u}^v\rangle| \\
&\qquad\qquad \lesssim \lambda^{\frac{\alpha+1}{2}}\|((\chi_\lambda\varphi_\lambda), \psi)\|^2_{L^\infty_x}\|\partial_x(\chi_\lambda(x/t)r^\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x} \\
&\qquad\qquad + \lambda^{\frac{\alpha+1}{2}}\|(\partial_x(\chi_\lambda\varphi_\lambda), \partial_x\psi)\|_{L^\infty_x}\|(\chi_\lambda\varphi_\lambda, \psi)\|_{L^\infty_x}\|\partial_x(\chi_\lambda(x/t)r^\lambda)\|_{L^\infty_x}\|\mathbf{u}^v\|_{L^1_x},
\end{aligned}$$

when $\alpha < 1$.

By interpolation, Moser's estimates for the fractional derivatives, along with Lemmas 4.29 and 4.30, and the condition $(t,v) \in \mathscr{D}$, it follows that the errors are acceptable. The L^2_x-bound is similar.

We seek to replace ψ by θ. We evaluate

$$\langle \partial_x P_\lambda \tilde{\chi}(Q(\psi) - Q(\theta)), \mathbf{u}^v\rangle$$

We have

$$|\tilde{\chi}(Q(\psi) - Q(\theta))| \lesssim \left|\tilde{\chi}\left(\frac{x - vt}{\sqrt{|ta''(\xi_v)|}}\right)\right| \int \frac{1}{|y|^{\alpha-1}}(|\delta^y\psi|^2 + |\delta^y\theta|^2)|\delta^y\beta_v^\lambda(x)|\, dy$$

The support condition of $\tilde{\chi}$ implies that x is in the region $|x - vt| \lesssim \delta x = t^{1/2}\lambda^{\frac{\alpha}{2}-1}$. From Lemma 4.32 we now get that

$$|\tilde{\chi}(Q(\psi) - Q(\theta))| \lesssim t^{-3/4}\lambda^{\frac{\alpha}{4}-\frac{1}{2}}\|\varphi\|_X \int \frac{1}{|y|^{\alpha-1}}(|\delta^y\psi|^2 + |\delta^y\theta|^2)\, dy$$

Bernstein's inequality and Lemma 4.6 imply that

$$\begin{aligned}|\partial_x P_\lambda\tilde{\chi}(Q(\psi) - Q(\theta))| &\lesssim t^{-3/4}\lambda^{\frac{\alpha}{4}+\frac{1}{2}}\|\varphi\|_X(\|\psi_x\|^2_{L^\infty_x} + \|\theta_x\|^2_{L^\infty})\|\mathbf{u}^v\|_{L^1_x} \\ &\quad + t^{-3/4}\lambda^{\frac{\alpha}{4}+\frac{1}{2}}\|\varphi\|_X(\|\psi\|_{L^\infty_x}\|\psi_x\|_{L^\infty_x} + \|\theta\|_{L^\infty_x}\|\theta_x\|_{L^\infty_x})\|\mathbf{u}^v\|_{L^1_x},\end{aligned}$$

when $\alpha > 1$, and

$$\begin{aligned}\lambda^{\frac{\alpha-1}{2}}|\partial_x P_\lambda\tilde{\chi}(Q(\psi) - Q(\theta))| &\lesssim t^{-3/4}\lambda^{\frac{3\alpha}{4}}\|\varphi\|_X(\|\psi\|^2_{L^\infty_x} + \|\theta\|^2_{L^\infty})\|\mathbf{u}^v\|_{L^1_x} \\ &\quad + t^{-3/4}\lambda^{\frac{3\alpha}{4}}\|\varphi\|_X(\|\psi\|_{L^\infty_x}\|\psi_x\|_{L^\infty_x} + \|\theta\|_{L^\infty_x}\|\theta_x\|_{L^\infty_x})\|\mathbf{u}^v\|_{L^1_x},\end{aligned}$$

when $\alpha < 1$.

From Lemmas 4.30 and 4.31, along with the condition $(t, v) \in \mathscr{D}$, it follows in both cases that this error is acceptable. The L^2_x-bound is similar.

We are left to analyze

$$t^{-3/2}\gamma(t, v)|\gamma(t, v)|^2 \left\langle \partial_x P_\lambda Q(\chi_\lambda e^{it\phi(x/t)}), \mathbf{u}^v \right\rangle.$$

Since by Lemma 4.28,

$$\|t^{-3/2}\gamma(t, v)|\gamma(t, v)|^2\|_{L^\infty_v(J_\lambda)} \lesssim \|\varphi_\lambda\|^3_{L^\infty_x}, \qquad \|t^{-3/2}\gamma(t, v)|\gamma(t, v)|^2\|_{L^2_v(J_\lambda)} \lesssim t^{-1/2}\|\varphi_\lambda\|^2_{L^\infty_x}\|\varphi_\lambda\|_{L^2_x}$$

it suffices to estimate

$$\begin{aligned}&|\langle\partial_x P_\lambda Q(\chi_\lambda e^{it\phi(x/t)}), \mathbf{u}^v\rangle - t^{\frac{1}{2}}q(\xi_v)\xi_v(\chi_\lambda(v))^3| \lesssim \lambda^{\max\{\frac{1-\alpha}{2}, 0\}-\delta}t^{\frac{1}{2}+C\epsilon^2}\epsilon \\ &|\langle\partial_x P_\lambda Q(\chi_\lambda e^{it\phi(x/t)}), \mathbf{u}^v\rangle - t^{\frac{1}{2}}q(\xi_v)\xi_v(\chi_\lambda(v))^3| \lesssim \lambda^{-\delta}t^{\frac{1}{2}+C\epsilon^2}\epsilon.\end{aligned}$$

We note that

$$\delta^y(\chi_\lambda e^{\pm it\phi(x/t)}) = \chi_\lambda\delta^y\left(e^{\pm it\phi(x/t)}\right) + \delta^y(\chi_\lambda)e^{\pm it\phi((x+y)/t)}.$$

Lemma 4.6, implies that

$$\left|\left\langle \partial_x P_\lambda \int \frac{1}{|y|^{\alpha-1}}\delta^y(\chi_\lambda)e^{it\phi((x+y)/t)}\delta^y(\chi_\lambda e^{-it\phi(x/t)})\delta^y(\chi_\lambda e^{it\phi(x/t)})\, dy, \mathbf{u}^v\right\rangle\right| \lesssim t^{-1/2}(\lambda^{3-\alpha} + \lambda^{4-\alpha}),$$

when $\alpha > 1$, and

$$\left|\left\langle \partial_x P_\lambda \int \frac{1}{|y|^{\alpha-1}} \delta^y(\chi_\lambda) e^{it\phi((x+y)/t)} \delta^y(\chi_\lambda e^{-it\phi(x/t)}) \delta^y(\chi_\lambda e^{it\phi(x/t)})\, dy, \mathbf{u}^v \right\rangle\right|$$
$$\lesssim t^{-1/2}(\lambda^{2-\alpha} + \lambda^{4-\alpha}),$$

when $\alpha < 1$.

The most problematic contribution in the case $\alpha > 1$ is the one that arises from the first term. We have

$$\lambda^\delta \lambda^{3-\alpha} t^{-1/2} \|\varphi_\lambda\|^3_{L^\infty_x} \lesssim \lambda^{-\alpha} t^{-2} \|\varphi\|^3_X \lesssim t^{-1-\delta} \epsilon^3 t^{C\epsilon^2}$$

The other term is analogous.

The most problematic contribution in the case $\alpha < 1$ is the one that arises from the second term. We have

$$\lambda^{\frac{\alpha+1}{2}+\delta} \lambda^{3-\alpha} t^{-1/2} \|\varphi_\lambda\|^3_{L^\infty_x} \lesssim t^{-\frac{3}{2}} \|\varphi\|^3_X \lesssim t^{-1-\delta} \epsilon^3 t^{C\epsilon^2}$$

The other term is analogous.

The L^2_v-bound is treated similarly, and so is the case in which one chooses the term $\delta^y(\chi_\lambda) e^{-it\phi((x+y)/t)}$ in the expansion of $\delta^y(\chi_\lambda e^{-it\phi(x/t)})$. This leaves us with

$$\langle \partial_x P_\lambda \left(\chi_\lambda(x/t)^3 Q(e^{it\phi(x/t)}) \right), \mathbf{u}^v \rangle$$

Lemma 4.35, implies that we can replace the latter with

$$\langle \partial_x P_\lambda \left(\chi_\lambda(x/t)^3 e^{it\phi(x/t)} (\phi'(x/t))^2 q(1) \right), \mathbf{u}^v \rangle,$$

with error bounded by

$$\lambda t^{1/2} \left(\frac{\lambda^{6-3\alpha} + \lambda^{5-2\alpha}}{t^{2-(4-\alpha)a}} + \frac{\lambda^{3-\alpha}}{t^{1-(2-\alpha)a}} + \frac{1}{t^{(1+\alpha)a}} \right)$$

when $\alpha > 1$, and

$$\lambda t^{1/2} \left(\frac{\lambda^{4-2\alpha}}{t^{2-(3-\alpha)a}} + \frac{\lambda^{3-\alpha}}{t^{1-(2-\alpha)a}} + \frac{1}{t^{(1+\alpha)a}} \right)$$

when $\alpha < 1$.

We note that one problematic contribution is the one arising from the last term. From $(t, v) \in \mathscr{D}$, we have the bound

$$\frac{\lambda^{1+\delta}}{t^{(1+\alpha)a-1/2}} \|\varphi_\lambda\|^3_{L^\infty_x} \lesssim \lambda^{-2} t^{-(1+\alpha)a-1} \|\varphi\|^3_X \lesssim \lambda^{-1-\delta} t^{-1-\delta} \epsilon^3 t^{C\epsilon^2}$$

We note that this contribution is acceptable. We now take $a = \frac{1}{10}$. The only other problematic contribution is the one arising from the first term, for which we bound

$$\begin{aligned}\lambda^{\delta}\frac{\lambda^{7-3\alpha}+\lambda^{6-2\alpha}}{t^{3/2-(4-\alpha)a}}\|\varphi_\lambda\|^3_{L^\infty_x} &\lesssim (\lambda^{4-3\alpha-\delta}+\lambda^{3-2\alpha-\delta})t^{(4-\alpha)a-3}\|\varphi\|^3_X\\ &\lesssim (\lambda^{4-3\alpha-\delta}+\lambda^{3-2\alpha-\delta})t^{-1-\delta}\epsilon^3 t^{C\epsilon^2},\end{aligned}$$

when $\alpha > 1$, and

$$\lambda^{2\alpha-3+\delta}\frac{\lambda^{5-2\alpha}}{t^{3/2-(3-\alpha)a}}\|\varphi_\lambda\|^3_{L^\infty_x} \lesssim t^{(3-\alpha)a-3}\|\varphi\|^3_X \lesssim t^{-6/5}\epsilon^3 t^{C\epsilon^2},$$

when $\alpha < 1$ (in both cases, $a \in (0,1)$).

The contribution arising from the second term can be immediately bounded by

$$\frac{\lambda^{\min\{\frac{7-\alpha}{2},4-\alpha\}+\delta}}{t^{1/2-(2-\alpha)a}}\|\varphi_\lambda\|^3_{L^\infty_x} \lesssim t^{(2-\alpha)a-\frac{3}{2}}\|\varphi\|^3_X \lesssim t^{-1-\delta}\epsilon^3 t^{C\epsilon^2}$$

The L^2_v-bound is similar. This means that we have to analyze

$$\begin{aligned}q(1)\langle\partial_x(\chi_\lambda(x/t)^3(\phi'(x/t))^2e^{it\phi(x/t)}),\mathbf{u}^v_\lambda\rangle &= q(1)\langle(\chi_\lambda(x/t)\phi'(x/t))^3e^{it\phi(x/t)},\mathbf{u}^v_\lambda\rangle\\ &+ q(1)t^{-1}\langle(3\chi_\lambda(x/t)^2\chi'_\lambda(x/t)(\phi'(x/t))^2+2\chi_\lambda(x/t)^3\phi'(x/t)\phi''(x/t)e^{it\phi(x/t)}),\mathbf{u}^v_\lambda\rangle,\end{aligned}$$

where the last contribution can be immediately shown to be an acceptable error by using the condition $(t,v)\in\mathscr{D}$. Further, we may replace $\mathbf{u}^v_\lambda$ by $\mathbf{u}^v$. To see this, from the proof of Lemma 5.8 in [86], we have

$$|P_{\neq\lambda}\mathbf{u}^v| \lesssim \lambda^{1-\frac{\alpha}{2}}(1+|y|)^{-1-\delta}t^{-1-\delta}\lambda^{-(1+\delta)\alpha}, \qquad y=(x-vt)|ta''(\xi_v)|^{-\frac{1}{2}},$$

and

$$|(\chi_\lambda(x/t)\phi'(x/t))^3e^{it\phi(x/t)}| \lesssim \lambda^3.$$

Here, we also recall that when $v\in J_\lambda$, $|\phi'(v)|\approx\lambda$. Thus,

$$\left|\left\langle(\chi_\lambda(x/t)\phi'(x/t))^3e^{it\phi(x/t)},P_{\neq\lambda}\mathbf{u}^v\right\rangle\right| \lesssim \lambda^3\lambda^{1-\frac{\alpha}{2}}\lambda^{-(1+\delta)\alpha}t^{-1-\delta}t^{1/2}\lambda^{\alpha/2-1} \lesssim t^{-1/2-\delta}\lambda^{3-(1+\delta)\alpha},$$

which along with the condition $(t,v)\in\mathscr{D}$ shows that this is an acceptable error.

As $\mathbf{u}^v$ is supported in the region $\left|\dfrac{x}{t}-v\right| \lesssim t^{-1/2}\lambda^{\frac{\alpha}{2}-1}$, we can replace x/t by v in $\chi_\lambda(x/t)\phi'(x/t)$, with acceptable errors. As $\chi_\lambda(v)=1$, the remaining term is now

$$iq(1)(\chi_\lambda(v)\xi_v)^3\left\langle e^{it\phi(x/t)},\mathbf{u}^v\right\rangle = t^{\frac{1}{2}}iq(\xi_v)\xi_v,$$

as desired. □

4.36.1 Closing the bootstrap argument

We recall that

$$\|\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-1+\delta+\delta_1}\|\varphi\|_X \lesssim \frac{1}{\sqrt{t}}\lambda^{-1+\delta+\delta_1}\epsilon t^{C\epsilon^2}$$

and when $\lambda > 1$,

$$\|\varphi_\lambda\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\lambda^{-(1+\alpha/2+2\delta)}\|\varphi\|_X \lesssim \frac{1}{\sqrt{t}}\lambda^{-(1+\alpha/2+2\delta)}\epsilon t^{C\epsilon^2}.$$

Thus, if $t \lesssim \lambda^N$ when $\lambda > 1$, and if $t \lesssim \lambda^{-N}$ when $\lambda \le 1$, where N can be chosen arbitrarily, we get the desired bounds. We are left to analyze $t \gtrsim \lambda^N$ when $\lambda > 1$, and $t \gtrsim \lambda^{-N}$ when $\lambda \le 1$.

We recall the following bounds in the elliptic region:

$$\||D_x|^{1-\delta-\delta_1}((1-\chi_\lambda)\varphi_\lambda(x))\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{3}{2}-\alpha-\delta-\delta_1}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{\frac{3}{2}-\alpha-\delta-\delta_1}+\lambda^{1-\alpha-\delta-\delta_1}}{t}\epsilon t^{C\epsilon^2}$$

$$\||D_x|^{\frac{\alpha}{2}+\delta}\partial_x((1-\chi_\lambda)\varphi_\lambda(x))\|_{L^\infty_x} \lesssim \frac{\lambda^{\frac{3-\alpha}{2}+\delta}}{t}(\|\varphi\|_X + \|\varphi_\lambda\|_{L^2_x}) \lesssim \frac{\lambda^{\frac{3-\alpha}{2}+\delta}+\lambda^{\frac{2-\alpha}{2}+\delta}}{t}\epsilon t^{C\epsilon^2}$$

which gives the desired bounds when $t \gtrsim \lambda^N$ ($\lambda > 1$), and $t \gtrsim \lambda^{-N}$ ($\lambda \le 1$). We still have to bound $\chi_\lambda\varphi_\lambda$. We recall that, if $x/t \in J_\lambda$, and $r(t,x) = \chi_\lambda\varphi_\lambda(t,x) - \frac{1}{\sqrt{t}}\chi_\lambda\gamma(t,x/t)e^{it\phi(x/t)}$,

$$t^{1/2}\|r^\lambda\|_{L^\infty_x} \lesssim t^{-1/4}\lambda^{\frac{\alpha}{4}-\frac{3}{2}}(1+\lambda^{-1})\epsilon t^{C\epsilon^2}$$

We note that

$$t^{-1/4}\lambda^{\frac{\alpha}{4}-\frac{3}{2}}(1+\lambda^{-1})\epsilon t^{C\epsilon^2} \lesssim \lambda^{1-\delta-\delta_1}\epsilon$$

when $\lambda \le 1$, because this is equivalent to

$$\lambda^{\frac{\alpha}{4}-\frac{5}{2}+\delta+\delta_1}(1+\lambda^{-1}) \lesssim t^{1/4-C\epsilon^2}$$

(this is true when $t \gtrsim \lambda^{-N}$), and that

$$t^{-1/4}\lambda^{\frac{\alpha}{4}-\frac{3}{2}}(1+\lambda^{-1})\epsilon t^{C\epsilon^2} \lesssim \lambda^{-\left(1+\frac{\alpha}{2}+3\delta/2\right)}\epsilon$$

when $\lambda > 1$, because this is equivalent to

$$\lambda^{\frac{3\alpha}{4}-\frac{1}{2}+3\delta/2}(1+\lambda^{-1}) \lesssim t^{1/4-C\epsilon^2}$$

(this is true when $t \gtrsim \max\{1, \lambda^N\}$).

This means that we only need the bounds

$$|\gamma(t,v)| \lesssim \epsilon\lambda^{-(1-\delta-\delta_1)}$$

when $\lambda \le 1$, and

$$|\gamma(t,v)| \lesssim \epsilon\lambda^{-\left(1+\frac{\alpha}{2}+3\delta/2\right)}$$

when $\lambda > 1$. By initializing at time $t = 1$, up to which the bounds are known to be true from the energy estimates, and by using Proposition 4.36, we reach the desired conclusion.

4.37 Modified scattering

In this section, we prove the final part of Theorem 4.1.2, which refers to the modified scattering behavior of the solutions constructed in Section 4.25.

As was shown by Hunter-Shu-Zhang [78] when $\alpha < 1$, the mass of the solutions to (4.1.2) stays conserved. We begin with a short proof of this property for the full range $(0, 2) - \{1\}$:

Proposition 4.38. For solutions φ of (4.1.2), $\|\varphi(t)\|_{L^2}$ is conserved in time.

Proof. We have

$$\begin{aligned}\frac{d}{dt}\|\varphi\|_{L^2_x}^2 &= \int \varphi_t \cdot \varphi \, dx \\ &= -2 \int \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi) |\delta|^y \varphi_x \, dy \cdot \varphi \, dx - 2c(\alpha) \int \varphi \cdot |\partial_x|^{\alpha-1} \varphi_x \, dx \\ &= -2 \int \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi) |\delta|^y \varphi_x \, dy \cdot \varphi \, dx := -2I.\end{aligned}$$

We note that by the change of variables $(x, y) \mapsto (x + y, -y)$,

$$I = - \int \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi) |\delta|^y \varphi_x \cdot \varphi(x + y) \, dx \, dy.$$

Thus,

$$\begin{aligned}-2I &= \int \int \frac{1}{|y|^{\alpha-1}} F(\delta^y \varphi) |\delta|^y \varphi_x \cdot (\varphi(x + y) - \varphi(x)) \, dx \, dy \\ &= \int |y|^{2-\alpha} \int F(\delta^y \varphi) \delta^y \varphi \cdot \delta^y \varphi_x \, dx \, dy \\ &= \int |y|^{2-\alpha} \int \partial_x (G(\delta^y \varphi)) \, dx \, dy = 0,\end{aligned}$$

where $G(x) = \dfrac{x^2}{2} - \dfrac{1}{2 - \alpha}(1 + x^2)^{1-\frac{\alpha}{2}}$. □

Recall the asymptotic equation

$$\dot{\gamma}(t, v) = iq(\xi_v)\xi_v t^{-1} |\gamma(t, v)|^2 \gamma(t, v) + f(t, v),$$

As $t \to \infty$, $\gamma(t, v)$ converges to the solution of the equation

$$\dot{\tilde{\gamma}}(t, v) = iq(\xi_v)\xi_v t^{-1} \tilde{\gamma}(t, v) |\tilde{\gamma}(t, v)|^2,$$

whose solution is

$$\tilde{\gamma}(t, v) = W(v) e^{iq(\xi_v)\xi_v \ln(t) |W(v)|^2}$$

We can immediately see that $W(v)$ is well-defined, as $|W(v)| = |\tilde{\gamma}(t,v)|$, which is a constant, and

$$W(v) = \lim_{s\to\infty} \tilde{\gamma}(e^{2s\pi/(q(\xi_v)\xi_v|W(v)|^2)}, v).$$

Here, the limit is taken over the positive integers.

Corollary 4.39. Let $v \in J_\lambda$. Under the assumption $(t,v) \in \mathscr{D}$, we have

$$\|\varphi_0\|_X \lesssim \epsilon \ll 1,$$

as well as $t \gtrsim \lambda^{-N}$ when $\lambda \le 1$, we have the asymptotic expansions

$$\|\gamma(t,v) - W(v)e^{iq(\xi_v)\xi_v \log t|W(v)|^2}\|_{L^\infty(J_\lambda)} \lesssim \lambda^{-\delta} g(\lambda,\alpha) t^{-\delta + C^2\epsilon^2}\epsilon. \tag{4.39.1}$$

$$\|\gamma(t,v) - W(v)e^{iq(\xi_v)\xi_v \log t|W(v)|^2}\|_{L^2(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-3/2}) t^{-\delta + C^2\epsilon^2}\epsilon. \tag{4.39.2}$$

Proof. This is an immediate consequence of Proposition 4.36. □

Proposition 4.40. Under the assumption

$$\|\varphi_0\|_X \lesssim \epsilon \ll 1,$$

the asymptotic profile W defined above satisfies

$$\|(-v)^{\frac{1+\delta}{\alpha-1}}(-v)^{\frac{\operatorname{sgn}(\log|\xi_v|)}{\alpha-1}(1+\delta/2)}|D_v|^{1-C_1\epsilon^2} W(v)\|_{L^2_v} \lesssim \epsilon$$

Moreover, when $s_0 = 0$, we also have $\|W(v)\|_{L^2_v} \lesssim \epsilon$.

Proof. We fix λ, and let $t \gtrsim \max\{1, \lambda^{-\alpha}\} := t_\lambda$. From Corollary 4.39 we know that

$$\|W(v) - e^{-iq(\xi_v)\xi_v \log t|\gamma(t,v)|^2}\gamma(t,v)\|_{L^2_v(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-3/2}) t^{-\delta + C^2\epsilon^2}\epsilon$$

From the product and chain rules with Lemma 4.28, we have

$$\left\|\partial_v\left(e^{-iq(\xi_v)\xi_v \log t|\gamma(t,v)|^2}\gamma(t,v)\right)\right\|_{L^2_v(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-2})\log(t)\epsilon t^{C^2\epsilon^2}.$$

In this case,

$$W(v) = O_{\dot{H}^1_v(J_\lambda)}(\lambda^{-\delta}(1+\lambda^{-2})\log(t)\epsilon t^{C^2\epsilon^2}) + O_{L^2_v(J_\lambda)}(\lambda^{-\delta}(1+\lambda^{-3/2}) t^{-\delta + C^2\epsilon^2}\epsilon), \qquad t \gtrsim t_\lambda.$$

By interpolation this will imply that for C_1 large enough we have

$$\|W(v)\|_{\dot{H}^{1-C_1\epsilon^2}_v(J_\lambda)} \lesssim \lambda^{-\delta}(1+\lambda^{-2})\epsilon.$$

By dyadic summation over $\lambda \ge 1$ and $\lambda \le 1$,

$$\|(-v)^{\frac{1+\delta}{\alpha-1}}(-v)^{\frac{\operatorname{sgn}(\log|\xi_v|)}{\alpha-1}(1+\delta/2)}|D_v|^{1-C_1\epsilon^2} W(v)\|_{L^2_v} \lesssim \epsilon$$

The last part (the case $s_0 = 0$) immediately follows from the conservation of mass. □

4.41 Euler fronts

Here we will briefly discuss the ingredients that are necessary in proving the local well-posedness result in the case of Euler fronts, $\alpha = 0$. For the purpose of this section, we define our control parameter B as $B := \|\varphi_x\|_{C^{0,\delta}}$. In what follows, $\mathbf{H}$ shall denote the Hilbert transform.

Lemma 4.42. We have

$$Q(\varphi, v) = R(x, D)v$$

where

$$\|(\partial_x R)(x, D)v\|_{L^2} \lesssim_A B^2.$$

Proof. We write

$$Q(\varphi, v) = \int F(\delta^y \varphi) \cdot (v(x+y) - v(x))\, dy \tag{4.42.1}$$

and set

$$r(x, \xi) = -\int F(\delta^y \varphi)(e^{i\xi y} - 1)\, dy.$$

We have

$$\partial_x r(x, \xi) = -\int F'(\delta^y \varphi)\delta^y \varphi_x (e^{i\xi y} - 1)\, dy,$$

hence by Lemma 4.6 ,

$$|\partial_x r| \lesssim_A \int |\delta^y \varphi||\delta^y \varphi_x|\, dy \lesssim_A B^2.$$

□

We now establish energy estimates for the paradifferential equation (4.11.2). We define the usual energy

$$E(v) := \int v^2\, dx.$$

Proposition 4.43. We have

$$\frac{d}{dt}E(v) \lesssim_A B^2 \|v\|_{L^2}^2 + \|f\|_{L^2}\|v\|_{L^2}. \tag{4.43.1}$$

Proof. Without loss of generality we assume $f = 0$. Using the equation (4.11.2) for v, we have

$$\begin{aligned}\frac{d}{dt}E^0_{\text{lin}}(v) &= 2\int v_t \cdot v\, dx = \int (\mathbf{H}v + 2\partial_x T_R)v \cdot v\, dx \\ &= \int T_{\partial_x R} v \cdot v\, dx\end{aligned}$$

This can be immediately seen to satisfy the desired estimate.

□

We now proceed to establish higher order energy estimates.

Proposition 4.44. Let $s \geq 0$. Given v solving (4.16.1), there exists a normalized variable v^s such that

$$\partial_t v^s - \partial_x Q(\varphi, v^s) - \frac{1}{2}\mathbf{H}v^s = f + \mathscr{R}(\varphi, v),$$

with

$$\|v^s\|_{L^2_x} \approx \||D_x|^s v\|_{L^2_x}$$

and $\mathscr{R}(\varphi, v)$ satisfying balanced cubic estimates,

$$\|\mathscr{R}(\varphi, v)\|_{L^2} \lesssim_A B^2 \|v\|_{L^2}. \tag{4.44.1}$$

Proof. Let v satisfy (4.16.1), where without loss of generality, $f = 0$. Then, we can immediately see that $v^s := |D_x|^s v$ satisfies

$$\partial_t v^s - \frac{1}{2}\mathbf{H}v - \partial_x T_R v^s = \mathscr{R}. \tag{4.44.2}$$

□

We thus obtain the following energy estimate:

Proposition 4.45. Let $s \geq 0$. There exist energy functionals $E^s(\varphi)$ such that we have the following:

a) Norm equivalence:

$$E^s(\varphi) \approx \|\varphi\|^2_{\dot{H}^s_x}$$

b) Energy estimates:

$$\frac{d}{dt}E^s(\varphi) \lesssim B^2 \|\varphi\|^2_{\dot{H}^s_x}$$

Proof. Let $E^s(\varphi) = E(v^s)$, where $E(v)$ is defined in Proposition 4.43, and v^s is defined in Proposition 4.44. Part a) is immediate, whereas part b) follows from Proposition 4.43. □

We also have the linearized counterpart:

Proposition 4.46. There exists an energy functional $E^{\text{lin}}(v)$ such that we have the following:

a) Norm equivalence:

$$E^{\text{lin}}(v) \approx \|v\|^2_{L^2_x}$$

b) Energy estimates:

$$\frac{d}{dt}E^{\text{lin}}(v) \lesssim B^2 \|v\|^2_{L^2_x}$$

Proof. Let $E^{\text{lin}}(v) = E(v)$, where $E^0(v)$ is defined in Proposition 4.43. Part a) is immediate, whereas part b) follows from Proposition 4.43. $\square$

Now the local well-posedness result easily follows as in the other cases.

Chapter 5

The dispersive Hunter-Saxton equation

5.1 Introduction

This chapter concerns the contents of [5]. We consider the Cauchy problem for the dispersive Hunter-Saxton equation

$$\begin{cases} u_t + uu_x + u_{xxx} = \dfrac{1}{2}\partial_x^{-1}(u_x^2), \\ u(0) = u_0, \end{cases} \tag{5.1.1}$$

where u is a real-valued function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$. Due to the Galilean invariance of (5.1.1), we may fix a definition for ∂_x^{-1},

$$\partial_x^{-1} f(x) = \int_{-\infty}^{x} f(y)\, dy,$$

where $f \in L_x^1(\mathbb{R})$.

The dispersive Hunter-Saxton equation is a perturbation of the Hunter-Saxton equation

$$u_t + uu_x = \frac{1}{2}\partial_x^{-1}(u_x^2), \tag{5.1.2}$$

which was introduced in [73] as an asymptotic model for the formation of nematic liquid crystals under a director field. The Hunter-Saxton equation (5.1.2) is completely integrable [79, 18] with a bi-Hamiltonian structure [118]. In the periodic case, local well-posedness and blow up phenomena were studied in [73, 145], while global weak solutions were studied in [24, 25]. For the non-periodic case, the Cauchy problem and blow up were studied in [144].

The Hunter-Saxton equation is also the high frequency limit of the Camassa-Holm equation,

$$(1 - \partial_x^2)u_t = 3uu_x - 2u_x u_{xx} - uu_{xxx}.$$

The local well-posedness and ill-posedness of the Camassa-Holm equation were studied in [38, 43, 63]. The global existence of strong solutions and blow up phenomena were investigated in [35, 37, 36, 38].

The dispersive Hunter-Saxton equation (5.1.1) first appeared in [80] as a dispersive regularization of (5.1.2). Complete integrability was later observed in [52].

In this chapter and in [5], we initiate the study of the well-posedness for the dispersive Hunter-Saxton equation (5.1.1). Throughout, we denote

$$X^s = L^\infty_x \cap \dot{H}^1_x \cap \dot{H}^{1+s}_x,$$

where $s \in [0,1]$. For brevity, we denote $X = X^1$. Our first pair of results concerns the well-posedness of (5.1.1) in X. We begin with local well-posedness:

Theorem 5.1.1. The dispersive Hunter-Saxton equation (5.1.1) is locally well-posed in X. Precisely, for every $R > 0$, there exists $T = T(R) > 0$ such that for every $u_0 \in X$ with $\|u_0\|_X < R$, the Cauchy problem (5.1.1) has a unique solution $u \in C([0,T], X)$. Moreover, the solution map $u_0 \mapsto u$ from X to $C([0,T], X)$ is continuous.

In both the dispersive and nondispersive cases of the Hunter-Saxton equation, the key difficulty is that the forcing term $\frac{1}{2}\partial_x^{-1}(u_x^2)$ is unbounded in any L^p space if $p < \infty$, and in particular, in L^2. As a result, it is necessary to consider the problem assuming only pointwise L^∞ control on u, similar to the analysis in [144] for the nondispersive case (5.1.2).

Further, although (5.1.1) superficially has a KdV-like dispersive term, the lack of spatial decay on the potential in front of the nonlinearity on the left-hand side of (5.1.1) obstructs direct access to dispersive tools, including local smoothing estimates. The consequence of this is that (5.1.1) exhibits quasilinear behavior, even in the presence of the KdV-like dispersive term. In particular, we strongly expect the solutions to exhibit only continuous dependence on initial data, although we do not investigate the failure of Lipschitz dependence in this chapter.

On the other hand, the dispersion in (5.1.1) introduces new subtleties to the behavior of solutions. Our second result states that, in contrast to the non-dispersive case (5.1.2), in which blow up can occur, the KdV-like term ensures that the problem is globally well-posed:

Theorem 5.1.2. The Cauchy problem (5.1.1) is globally well-posed in X. Moreover, for every $t \geq 0$, we have the global in time bounds

$$\begin{aligned}
\|u(t)\|_{L^\infty_x} &\lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}),\\
\|u(t)\|^2_{\dot{H}^2_x} &\lesssim \|u_0\|^2_{\dot{H}^2_x} + \|u_0\|_{X^0} E_1 + t(E_1 + E_1^{1/2})E_1,
\end{aligned}$$

where $E_1 = \|u_0\|^2_{\dot{H}^1_x}$ is given by the first conserved energy (5.1.3) below.

We remark that the L^∞ estimate holds even for solutions which are only in $X^0 = L^\infty \cap \dot{H}^1$.

Our proof of Theorem 5.1.1 follows a bounded iterative scheme which treats separately the high and low frequency components. To prove continuous dependence on initial data in our quasilinear setting, we use frequency envelopes, introduced by Tao in [135]. A systematic presentation of the use of frequency envelopes in the study of local well-posedness theory for quasilinear problems can be found in the expository paper [87], which we broadly follow in the the proof of Theorem 5.1.1.

Theorem 5.1.2 will follow from the conservation of the quantities $E_1(t)$ and $E_2(t)$ under evolution by (5.1.1),

$$\begin{aligned} E_1(t) &= \int_{\mathbb{R}} u_x(t)^2\, dx, \\ E_2(t) &= \int_{\mathbb{R}} u_{xx}(t)^2 - u(t)u_x(t)^2\, dx. \end{aligned} \tag{5.1.3}$$

Using the X^1 well-posedness of Theorems 5.1.1 and 5.1.2 as a starting point, our next pair of results extend the well-posedness to lower regularity initial data:

Theorem 5.1.3. For each $s \in (\frac{1}{2}, 1)$, the Cauchy problem (5.1.1) is locally well-posed in X^s.

The local well-posedness of Theorem 5.1.3 is in the same sense as that of Theorem 5.1.1. Here, we leverage Theorem 5.1.1 to construct X^s solutions as limits of sequences of smooth solutions, by proving an estimate for differences of solutions to establish convergence. This in turn is a consequence of an estimate for the linearized equation associated to (5.1.1),

$$w_t + (uw)_x + w_{xxx} = \partial_x^{-1}(u_x w_x). \tag{5.1.4}$$

Theorem 5.1.4. For each $s \in (\frac{1}{2}, 1)$, the Cauchy problem (5.1.1) is globally well-posed in X^s. Moreover, for every $t \geq 0$,

$$\begin{aligned} \|u(t)\|_{L^\infty_x} &\lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}) \\ \|u(t)\|^2_{\dot{H}^{1+s}_x} &\lesssim \langle t\rangle^4 E_1^2 \langle \|u_0\|_{X^0} + E_1\rangle^2 + \|u_0\|^2_{\dot{H}^{1+s}_x}, \end{aligned} \tag{5.1.5}$$

where $E_1 = \|u_0\|^2_{\dot{H}^1_x}$.

To prove Theorem 5.1.4, we construct modified energy functionals which are norm-equivalent to $\dot{H}^{1+s}$, based on the quadratic normal form transformation associated to (5.1.1). The use of a normal form analysis to study low regularity solutions has appeared a few times in the literature, although the precise implementation has gone under various guises, and taken various names. In the context of the KdV equation, $H^{-3/4}$ solutions were constructed by Christ-Colliander-Tao [30] using a generalized Miura transform. In the context of the Benjamin-Ono equation, Ifrim-Tataru [85] constructed L^2 solutions using a partial normal form transform, combined with an exponential renormalization.

In our current setting, due in particular to the quasilinear nature of (5.1.1), a direct normal form analysis is inaccessible as it would result in an unbounded normal form transformation. This is the motivation for using the approach of constructing modified energies, which is a robust alternative implementation of normal form analysis which is applicable even in the quasilinear setting. This approach was first introduced by Hunter-Ifrim-Tataru-Wong [72] in the context of the Burgers-Hilbert equation. It was further developed in the gravity water wave setting by Hunter-Ifrim-Tataru [70], which established almost-global well-posedness, and in the Benjamin-Ono setting by Ifrim-Tataru [85], which additionally established dispersive decay.

This chapter is organized as follows. In Section 5.2, we describe the notation we use throughout the chapter, including notation from the classical Littlewood-Paley trichotomy, as well as frequency envelopes. In Section 5.4, we present some existence results at various degrees of regularity for linear equations that arise throughout the proofs of the main results. In Section 5.10, using an iterative scheme, we prove the higher regularity local well-posedness result, while in Section 5.11, by using the conserved quantities E_1 and E_2, we show that the dispersive Hunter-Saxton equation (5.1.1) is globally well-posed.

Section 5.12 constructs and analyzes a modified energy based on the normal form associated to the Hunter-Saxton equation, in order to obtain bounds on the growth of the X^s-norm. Lastly, Section 5.15 discusses an estimate for the linearized equation (5.1.4), as well as one for differences of solutions. These results are then used to prove the low regularity local well-posedness result in Section 5.18.

5.2 Littlewood-Paley trichotomy and frequency envelopes

In this section, we introduce notations and some classical results from the Littlewood-Paley trichotomy and paradifferential calculus that we shall use in the sequel. We also discuss frequency envelopes, which we will use in particular to establish continuous dependence on initial data.

We use the standard Littlewood-Paley decomposition throughout the article. Precisely, let $\phi \in C_c^\infty(\mathbb{R})$ be an even function such that $\phi(\xi) = 1$ on $[-1, 1]$ and $\phi(\xi) = 0$ outside of the interval $[-2, 2]$. For an integer k and a Schwartz function f, we define

$$\widehat{P_{\leq k} f}(\xi) = \phi\left(\frac{\xi}{2^k}\right) \widehat{f}(\xi).$$

We also define $P_k f = P_{\leq k} f - P_{\leq k-1} f$ and $P_{>k} f = f - P_{\leq k} f$. We shall also use the alternate notations $P_{\leq k} f = f_{\leq k}$, $P_k f = f_k$, and $P_{>k} f = f_{>k}$.

We use D_x and $|D_x|^s$ to denote the standard Fourier multiplier operators. Precisely, $\widehat{D_x f}(\xi) = \xi \hat{f}(\xi)$ and $\widehat{|D_x|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$.

We will use the following Bernstein inequalities:

Theorem 5.2.1. Let $k \in \mathbb{Z}$, $l \in \mathbb{N}$, $1 \le p \le q \le \infty$, and $s > 0$. Then

$$\begin{aligned}
\||D_x|^s P_{\le k} f\|_{L^p_x} &\lesssim 2^{ks} \|P_{\le k} f\|_{L^p_x}, \\
\||D_x|^s P_k f\|_{L^p_x} &\simeq 2^{ks} \|P_k f\|_{L^p_x}, \\
\|P_{>k} f\|_{L^p_x} &\lesssim 2^{-ks} \||D_x|^s P_{>k} f\|_{L^p_x}, \\
\|P_{\le k} f\|_{L^q_x} &\lesssim 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_{\le k} f\|_{L^p_x}, \\
\|P_k f\|_{L^q_x} &\lesssim 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_k f\|_{L^p_x}.
\end{aligned}$$

We next record several classical paraproduct estimates. For two functions f_1 and f_2 belonging to some Lebesgue or Sobolev spaces, we write:

$$f_1 f_2 = \sum_{k\in\mathbb{Z}} (f_1)_{<k-4} (f_2)_k + \sum_{k\in\mathbb{Z}} (f_2)_{<k-4} (f_1)_k + \sum_{\substack{k,l\in\mathbb{Z}\\ |k-l|\le 4}} (f_1)_k (f_2)_l := T_{f_1} f_2 + T_{f_2} f_1 + \Pi(f_1, f_2).$$

Then we have the following Coifman-Meyer estimates:

Theorem 5.2.2. a) Let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ where $1 \le q, r < \infty$, $1 \le p \le \infty$, and $0 \le s < \infty$. Then

$$\||D_x|^s T_{f_1} f_2\|_{L^r_x} + \|T_{|D_x|^s f_1} f_2\|_{L^r_x} \lesssim \|f_1\|_{L^p_x} \||D_x|^s f_2\|_{L^q_x}.$$

b) Additionally let $s = s_1 + s_2 = s_1' + s_2'$, where $s_1, s_2, s_1', s_2' \in [0, \infty)$. Then

$$\begin{aligned}
\||D_x|^s \Pi(f_1, f_2)\|_{L^r_x} &\lesssim \||D_x|^{s_1} f_1\|_{L^p_x} \||D_x|^{s_2} f_2\|_{L^q_x} \\
\|\Pi(|D_x|^{s_1'} f_1, |D_x|^{s_2'} f_2)\|_{L^r_x} &\lesssim \||D_x|^{s_1} f_1\|_{L^p_x} \||D_x|^{s_2} f_2\|_{L^q_x}.
\end{aligned}$$

c) Let $1 < p < \infty$. Then

$$\begin{aligned}
\|T_{f_1} f_2\|_{L^p_x} &\lesssim \|f_1\|_{L^\infty_x} \|f_1\|_{L^p_x}, \\
\|T_{f_1} f_2\|_{L^p_x} &\lesssim \|f_1\|_{L^p_x} \|f_2\|_{BMO}, \\
\|\Pi(f_1, f_2)\|_{L^p_x} &\lesssim \|f_1\|_{BMO} \|f_2\|_{L^p_x}.
\end{aligned}$$

Proof. See [34], as well as Chapter 3.5 in [141], and Lemmas 3.2, 3.3, 6.2 and 6.3 in [133], □

We also have the following Kato-Ponce commutator estimate:

Theorem 5.2.3. Let $0 < s < 1$. Then

$$\begin{aligned}
\|[|D_x|^s, f_1](f_2)_x\|_{L^2_x} &\lesssim_s \|(f_1)_x\|_{L^\infty_x} \||D_x|^s f_2\|_{L^2_x}, \\
\|[\langle D_x\rangle^s, f_1](f_2)_x\|_{L^2_x} &\lesssim_s \|(f_1)_x\|_{L^\infty_x} \|\langle D_x\rangle^s f_2\|_{L^2_x}.
\end{aligned}$$

Proof. See [102] and Chapter 3.6 (estimate 3.6.2) in [141] for the inhomogeneous case. For the homogeneous case, see Lemma 6.2 in [133] . □

We note that the previous result is still true when we replace $|D_x|^s$ by any differential operator of order $0 < s < 1$.

We now define $s$harp frequency envelopes. This notion was introduced by Tao, and is broadly presented by Ifrim and Tataru in their expository paper [87].

Definition 5.3. Let $\delta > 0$ and $f \in L^2$. We say that $\{c_j\}_{j\in\mathbb{Z}}$ is a sharp frequency envelope for f if:

a) $\|f_k\|_{L^2} \lesssim c_k$ for every $k \in \mathbb{Z}$,

b) $\displaystyle\sum_k c_k^2 \sim \|f\|_{L^2}^2$,

c) and $\dfrac{c_j}{c_k} \lesssim 2^{\delta|j-k|}$ for every $j, k \in \mathbb{Z}$.

It is easy to verify that if $f \in L^2$, then

$$c_k = \sup_{j\in\mathbb{Z}} 2^{-\delta|j-k|}\|P_j f\|_{L^2}$$

is an L^2_x-sharp frequency envelope for f. It can be immediately seen that this definition can be extended to any L^2-based Sobolev space of the form H^s or $\dot{H}^s$, where $s \in \mathbb{R}$.

5.4 Linear analysis

The goal of this section is to collect results for the linear analysis which will be used in the sequel. We study the well-posedness of a linear equation which will be used in the iteration scheme for the proof of Theorem 5.1.1.

We first prove well-posedness and energy estimates for initial data in L^2.

Lemma 5.5. Let $T > 0$, $a, b \in L^\infty_t([0,T], \dot{W}^{1,\infty})$, $F \in L^1_t([0,T], L^2_x)$, $v_0 \in L^2_x$. Then the Cauchy problem

$$\begin{cases} v_t + av_x + b_x v + v_{xxx} = F \\ v(0) = v_0 \end{cases} \tag{5.5.1}$$

admits a unique solution $v \in L^\infty_t([0,T], L^2_x)$ which satisfies the energy estimate

$$\frac{d}{dt}\|v(t)\|^2_{L^2_x} \lesssim \|F(t)\|_{L^2_x}\|v(t)\|_{L^2_x} + (\|a_x(t)\|_{L^\infty_x} + \|b_x(t)\|_{L^\infty_x})\|v(t)\|^2_{L^2_x}.$$

Proof. Let us assume that v is a solution to the Cauchy problem. We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} v^2(t)\,dx &= 2\int_{\mathbb{R}} v(t)v_t(t)\,dx \\ &= 2\int_{\mathbb{R}} v(t)(F(t) - a(t)v_x(t) - b_x(t)v(t) - v_{xxx}(t))\,dx \\ &= 2\int_{\mathbb{R}} v(t)F(t)\,dx + \int_{\mathbb{R}} a_x(t)v^2(t)\,dx - 2\int_{\mathbb{R}} b_x(t)v^2(t)\,dx \\ &\lesssim \|v(t)\|_{L^2_x}\|F(t)\|_{L^2_x} + \|v(t)\|^2_{L^2_x}(\|a_x(t)\|_{L^\infty_x} + \|b_x(t)\|_{L^\infty_x}).\end{aligned} \tag{5.5.2}$$

We obtain the desired energy estimate, which also establishes uniqueness.

It remains to show existence, for which we follow a standard duality argument (for the general theory, see Theorem 23.1.2 in [67]). We first determine the adjoint problem. For an arbitrary w, a formal computation shows that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}(v_t + av_x + b_xv + v_{xxx})w\,dx\,dt = &\int_{\mathbb{R}} v(T)w(T)\,dx - \int_{\mathbb{R}} v(0)w(0)\,dx \\ &- \int_0^T\int_{\mathbb{R}}(w_t + aw_x + a_xw + b_xw + w_{xxx})v\,dx\,dt.\end{aligned}$$

We write $w_t + aw_x + (a_x + b_x)w + w_{xxx} = G$ and $w(T) = w_T$. Thus,

$$\int_0^T\int_{\mathbb{R}} Fw\,dx\,dt + \int_{\mathbb{R}} v_0w(0)\,dx = \int_{\mathbb{R}} v(T)w_T\,dx - \int_0^T\int_{\mathbb{R}} Gv\,dx\,dt$$

and we have the adjoint problem

$$\begin{cases} w_t + aw_x + (a_x + b_x)w + w_{xxx} = G \\ w(T) = w_T. \end{cases} \tag{5.5.3}$$

Using the energy estimate (5.5.2) of the original equation, we have

$$\|w(t)\|_{L^2_x} \lesssim \|w_T\|_{L^2_x} + \|G\|_{L^1_tL^2_x}.$$

In particular, we conclude that if the adjoint problem has a solution, then it is unique.

Let

$$\begin{aligned}Y = \{&(g,\tilde{G}) \in L^2_x \times L^1_tL^2_x([0,T]\times\mathbb{R}) \mid \\ &\text{there exists } h \in L^\infty_tL^2_x \text{ solving the adjoint problem with } (h_T, G) = (g,\tilde{G})\ \}.\end{aligned}$$

We define the functional $\alpha : Y \to \mathbb{R}$ by

$$\alpha(g,\tilde{G}) = \int_0^T\int_{\mathbb{R}} Fh\,dx\,dt + \int_{\mathbb{R}} v_0h(0)\,dx,$$

which is well-defined by uniqueness for the adjoint problem. It is also bounded, as

$$\begin{aligned}|\alpha(g,\tilde{G})| &\lesssim \|v_0\|_{L^2_x}\|h(0)\|_{L^2_x} + \|F\|_{L^1_tL^2_x}\|h\|_{L^\infty_tL^2_x} \\ &\lesssim \|v_0\|_{L^2_x}(\|g\|_{L^2_x} + \|\tilde{G}\|_{L^1_tL^2_x}) + \|F\|_{L^1_tL^2_x}(\|g\|_{L^2_x} + \|\tilde{G}\|_{L^1_tL^2_x}) \\ &\lesssim (\|v_0\|_{L^2_x} + \|F\|_{L^1_tL^2_x})(\|g\|_{L^2_x} + \|\tilde{G}\|_{L^1_tL^2_x}).\end{aligned}$$

Using the Hahn-Banach Theorem, we extend α to a functional β defined on $L^2_x \times L^1_tL^2_x$. This uniquely corresponds an element of $L^2_x \times L^\infty_tL^2_x$, whose second component is the desired solution v.

□

We extend the previous result to the case when the initial data is in H^1:

Lemma 5.6. Let $T > 0$, $a, b \in L^\infty_t\dot{W}^{1,\infty}_x$, $b \in L^\infty_t\dot{H}^2_x$, $F \in L^1_tH^1_x$, and $v_0 \in H^1_x$. Then the Cauchy problem (5.5.1) has a unique solution $v \in L^\infty_tH^1_x$ which satisfies the energy estimate

$$\frac{d}{dt}\|v\|^2_{\dot{H}^1_x} \lesssim (\|F\|_{\dot{H}^1_x} + \|b_{xx}\|_{L^2_x}\|v\|_{L^\infty_x})\|v\|_{\dot{H}^1_x} + (\|a_x\|_{L^\infty_x} + \|b_x\|_{L^\infty_x})\|v\|^2_{\dot{H}^1_x}.$$

In particular, if u is a solution of the dispersive Hunter-Saxton equation (5.1.1), then

$$\frac{d}{dt}\|u_x\|^2_{\dot{H}^1_x} \lesssim \|u_x\|_{L^\infty_x}\|u_x\|^2_{\dot{H}^1_x}.$$

Proof. We first consider the regularized equation

$$v_t + v_{xxx} + av_x + (b_{\le m})_xv = F.$$

By applying Lemma 5.5, we obtain a unique solution $v^m \in L^\infty_tL^2_x$. We first show that, in fact, $v^m \in L^\infty_tH^1_x$. To see this, note that v^m_x formally satisfies the following equation for $\tilde{v}$:

$$\tilde{v}_t + \tilde{v}_{xxx} + (a_x + (b_{\le m})_x)\tilde{v} + a\tilde{v}_x = F_x - (b_{xx})_{\le m}v^m \tag{5.6.1}$$

To apply Lemma 5.5 to (5.6.1), we check that the source $F_x - (b_{xx})_{\le m}v^m$ belongs to $L^\infty_tL^2_x$. Indeed,

$$\|F_x\|_{L^\infty_tL^2_x} < \infty$$

and

$$\begin{aligned}\|(b_{xx})_{\le m}v^m\|_{L^\infty_tL^2_x} &\le \|(b_{xx})_{\le m}\|_{L^\infty_{t,x}}\|v^m\|_{L^\infty_tL^2_x} \le 2^m\|(b_x)_{\le m}\|_{L^\infty_{t,x}}\|v^m\|_{L^\infty_tL^2_x} \\ &\le 2^m\|b_x\|_{L^\infty_{t,x}}\|v^m\|_{L^\infty_tL^2_x} < \infty.\end{aligned}$$

Thus, via Lemma 5.5, we obtain that (5.6.1) admits a unique solution $\tilde{v^m} \in L^\infty_tL^2_x$ so that $v^m_x = \tilde{v^m}$ and $v^m \in L^\infty_tH^1_x$ as desired. In particular, by Sobolev embedding, we also get that $v^m \in L^\infty_{t,x}$.

We remark that the crude estimate above on $(b_{xx})_{\leq m}v^m$ will not be used again in the proof; its sole purpose was to permit use of Lemma 5.5 on (5.6.1). We shall obtain uniform H^1_x-estimates for the sequence $\{v^m\}_{m\geq 0}$ in what follows.

Using (5.5.2), we find

$$\frac{d}{dt}\int_{\mathbb{R}}(v^m)^2\,dx \lesssim \|v^m\|_{L^2_x}\|F\|_{L^2_x} + (\|a_x\|_{L^\infty_x} + 2\|b_x\|_{L^\infty_x})\|v^m\|^2_{L^2_x}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(v^m_x)^2\,dx &\lesssim \|v^m_x\|_{L^2_x}\|F_x - (b_{xx})_{\leq m}v^m\|_{L^2_x} + \|a_x + 2(b_{\leq m})_x\|_{L^\infty_x}\|v^m_x\|^2_{L^2_x}\\ &\lesssim \|v^m_x\|_{L^2_x}\|F_x\|_{L^2_x} + \|(b_{xx})_{\leq m}\|_{L^2_x}\|v^m\|_{L^\infty_x}\|v^m_x\|_{L^2_x}\\ &\quad + (\|a_x\|_{L^\infty_x} + \|(b_x)_{\leq m}\|_{L^\infty_x})\|v^m_x\|^2_{L^2_x}\\ &\lesssim \|v^m_x\|_{L^2_x}\|F_x\|_{L^2_x} + \|b_{xx}\|_{L^2_x}\|v^m\|_{L^\infty_x}\|v^m_x\|_{L^2_x} + (\|a_x\|_{L^\infty_x} + \|b_x\|_{L^\infty_x})\|v^m_x\|^2_{L^2_x}.\end{aligned}$$

Denoting

$$E^m(t) = \int_{\mathbb{R}}(v^m(t))^2\,dx + \int_{\mathbb{R}}(v^m_x(t))^2\,dx,$$

we have

$$\frac{d}{dt}E^m(t) \lesssim (E^m(t))^{1/2}\|F(t)\|_{H^1_x} + (\|a_x(t)\|_{L^\infty_x} + \|b_x(t)\|_{L^\infty_x} + \|b_{xx}(t)\|_{L^2_x})E^m(t).$$

From Grönwall's lemma applied to $(E^m(t))^{1/2}$, we infer that

$$\begin{aligned}E^m(t) \leq\; & e^{\frac{C}{2}\int_0^T \|a_x(s)\|_{L^\infty_x} + \|b_x(s)\|_{L^\infty_x\cap \dot{H}^1_x}\,ds}\cdot\\ &\left(\|v_0\|_{H^1_x} + \int_0^T e^{-\frac{C}{2}\int_0^s \|a_x(\tau)\|_{L^\infty_x} + \|b_x(\tau)\|_{L^\infty_x\cap\dot{H}^1_x}\,d\tau}\|F(s)\|_{H^1_x}\,ds\right),\end{aligned}$$

uniformly in m and $t\in[0,T]$.

Let $l\geq 0$ and $z = v^{m+l} - v^m \in L^\infty_t L^2_x$. We see that z solves

$$z_t + z_{xxx} + az_x + (b_x)_{\leq m+l}z = -(b_x)_{m<\cdot\leq m+l}v^m =: H.$$

Let $e := \sup_{m\geq 1}\sup_{t\in[0,T]} E^m(t) < \infty$. We estimate the source term (by using Bernstein's inequalities and Sobolev embedding):

$$\begin{aligned}\|H\|_{L^\infty_t L^2_x} &\lesssim \|(b_x)_{m<\cdot\leq m+l}\|_{L^\infty_t L^2_x}\|v^m\|_{L^\infty_{t,x}} \lesssim 2^{-m}\|(b_{m<\cdot\leq m+l})_{xx}\|_{L^\infty_t L^2_x}\|v^m\|_{L^\infty_t H^1_x}\\ &\lesssim 2^{-m}\|b_{xx}\|_{L^\infty_t L^2_x}e^{1/2}.\end{aligned}$$

By applying the energy estimate provided by Lemma 5.5 with Grönwall, we obtain

$$\begin{aligned}\|z(t)\|_{L^2_x} &\le e^{\frac{C}{2}\int_0^t \|a_x(s)\|_{L^\infty_x}+2\|(b_x(s))_{\le m+l}\|_{L^\infty_x}\,ds}\\ &\quad\cdot\left(\frac{C}{2}\int_0^t e^{-\frac{C}{2}\int_0^s \|a_x(\tau)\|_{L^\infty_x}+2\|(b_x(\tau))_{\le m+l}\|_{L^\infty_x}\,d\tau}\|H(s)\|_{L^2_x}\,ds\right)\\ &\le \frac{C}{2}\int_0^t e^{\frac{C}{2}\int_s^t \|a_x(\tau)\|_{L^\infty_x}+2\|(b_x(\tau))_{\le m+l}\|_{L^\infty_x}\,d\tau}\|H(s)\|_{L^2_x}\,ds\\ &\le \frac{C}{2}\int_0^t e^{\frac{C}{2}\int_s^t \|a_x(\tau)\|_{L^\infty_x}+2\|b_x(\tau)\|_{L^\infty_x}\,d\tau}\|H(s)\|_{L^2_x}\,ds\\ &\lesssim T2^{-m}e^{1/2}\|b_{xx}\|_{L^\infty_t L^2_x}.\end{aligned}$$

Thus, v^m is a Cauchy sequence in $L^\infty_t L^2_x$, which means that it converges to a solution v. As v^m is bounded in $L^\infty_t H^1_x$, Lemma 5.9 implies $v \in L^\infty_t H^1_x$. The energy estimates of Lemma 5.5 also prove uniqueness. A similar computation to the one carried out for v^m provides the desired energy estimate. In particular, if u is a solution of (5.1.1), then u_x is a solution of (5.5.1) with $a = u$, $b = -u_x/2$, and $F = 0$, so that the desired estimate follows. □

Using this, we establish persistence of regularity for (5.1.1):

Lemma 5.7. Let $T > 0$, and $u \in C([0,T], X)$ a solution for the dispersive Hunter-Saxton equation (5.1.1). If $u(0) \in X \cap \dot{H}^{n+1}_x(\mathbb{R})$, then $u \in L^\infty_t([0,T], X \cap \dot{H}^{n+1}_x)$. Furthermore, in the case $n = 2$, we have the energy estimate

$$\frac{d}{dt}\|u_{xx}(t)\|^2_{H^1_x} \lesssim \|u_x(t)\|_{L^\infty_x}\|u_{xx}(t)\|^2_{H^1_x}.$$

Proof. Observe that u_{xx} formally satisfies

$$v_t + uv_x + 2u_x v + v_{xxx} = 0.$$

As $u \in L^\infty_t X$, by applying Lemma 5.6, we infer that the problem admits a unique solution $v \in L^\infty_t H^1_x$. In particular, v solves the problem in the sense of distributions, so that $v = u_{xx}$ and $u \in L^\infty_t(X \cap \dot{H}^3_x)$, along with the energy estimate, as desired.

For $n > 2$, observe that $\partial_x^n u$ formally satisfies

$$v_t + uv_x + 2u_x v + v_{xxx} = P(u_{xx}, ..., \partial_x^{n-2}u), \tag{5.7.1}$$

where P is a quadratic polynomial. The result follows by induction and Lemma 5.6. □

We now establish the following L^∞ estimate that will be used in the proof of several other results, including the iteration for the proof of Theorem 5.1.1:

Lemma 5.8. Let $T > 0$, $a \in L_t^\infty([0,T], W_x^{1,\infty})$, and $w \in L_t^\infty([0,T], L_x^\infty)$ satisfy

$$w_t + aw_x + w_{xxx} = f. \tag{5.8.1}$$

Then w satisfies

$$\frac{d}{dt}\|w_{\leq 0}(t)\|_{L_x^\infty} \lesssim \|w_{\leq 0}(t)\|_{L_x^\infty} + \|f_{\leq 0}(t)\|_{L_x^\infty} + \|a(t)\|_{W_x^{1,\infty}}\|w(t)\|_{L_x^\infty}$$

.

Proof. By applying the frequency projection $P_{\leq 0}$, we obtain

$$(w_{\leq 0})_t + (aw_x)_{\leq 0} + (w_{\leq 0})_{xxx} = f_{\leq 0}$$

and estimate

$$\begin{aligned}\|(aw_x + w_{xxx})_{\leq 0}\|_{L_x^\infty} &\lesssim \|((aw)_x - (a_x w))_{\leq 0}\|_{L_x^\infty} + \|(w_{\leq 0})_{xxx}\|_{L_x^\infty} \\ &\lesssim \|(aw)_{\leq 0}\|_{L_x^\infty} + \|a_x w\|_{L_x^\infty} + \|w_{\leq 0}\|_{L_x^\infty} \\ &\lesssim (\|a\|_{L_x^\infty} + \|a_x\|_{L_x^\infty})\|w\|_{L_x^\infty} + \|w_{\leq 0}\|_{L_x^\infty},\end{aligned}$$

using at the second line that derivatives falling on unit frequencies are bounded by a constant. □

Lastly, we observe a technical result which will be used in the proof of Theorem 5.1.1 to show that the solution of (5.1.1) has the desired regularity:

Lemma 5.9. Let $T > 0$ and $\{v^n\}_{n\geq 0} \in L_t^\infty([0,T], H_x^1)$ be a bounded sequence such that

$$v^n \to v \in L_t^\infty([0,T], L_x^2).$$

Then $v \in L_t^\infty([0,T], H_x^1)$.

Proof. Let $M > 0$ be such that $\|v^n\|_{L_t^\infty H_x^1} \leq M$ for every $n \geq 0$. Fix $t \in [0,T]$ such that $v^n(t)$ converges to $v(t)$ in $L_x^2(\mathbb{R})$, and $\|v^n(t)\|_{H_x^1} \leq M$. We omit t in the notations below.

As v^n is bounded in $H_x^1(\mathbb{R})$, which is a Hilbert space and hence reflexive, we infer that there exists a subsequence $\{v^{n_k}\}_{k\geq 0}$ that converges weakly to some $g \in H_x^1(\mathbb{R})$. In particular, v^{n_k} converges to g in the sense of distributions. On the other hand, v^n converges to v in $L_x^2(\mathbb{R})$ and in the sense of distributions, so $v = g \in H_x^1(\mathbb{R})$.

Let $w \in H_x^1(\mathbb{R})$ with $\|w\|_{H_x^1} = 1$. As $\|v^{n_k}\|_{H_x^1} \leq M$, we have $|\langle v^{n_k}, w\rangle| \leq \|v^{n_k}\|_{H_x^1}\|w\|_{H_x^1} \leq M$, hence

$$|\langle v, w\rangle| = \lim_{k\to\infty} |\langle v^{n_k}, w\rangle| \leq M.$$

We infer that $\|v\|_{H_x^1} \leq M$. This finishes the proof. □

5.10 Local well-posedness

In this section we prove Theorem 5.1.1.

Let $C > 0$ be a large absolute constant which may vary from line to line, and let $T > 0$ be a small number to be fixed later. Let $\|u_0\|_X < R$. We inductively define a sequence $\{u^n\}_{n\geq 0} \in L_t^\infty([0,T], X)$. For $n = 0$ we set $u^0(t,x) = u_0(x)$. For $n > 0$, we will set $u^{n+1} \in L_t^\infty([0,T], X)$ as the unique solution of the Cauchy problem

$$\begin{cases} u_t^{n+1} + u_{xxx}^{n+1} + u^n u_x^{n+1} = \dfrac{\partial_x^{-1}((u_x^n)^2)}{2}, \\ u^{n+1}(0) = u_0. \end{cases} \tag{5.10.1}$$

5.10.1 Existence and uniform bounds for (5.10.1)

Here we show existence and estimates for (5.10.1) in $L_t^\infty([0,T], X)$.

Existence for u^{n+1} in $L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)$

We first show that (5.10.1) has a solution $u^{n+1} \in L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)$ with

$$E^{n+1}(t) := \int_{\mathbb{R}} (u_{xx}^{n+1}(t))^2 + (u_x^{n+1}(t))^2 \, dx \leq K\|u_0\|_X^2 =: E,$$

for $K > 0$ a large absolute constant. We assume by induction that this is true for u^n.

We consider the Cauchy problem

$$\begin{cases} v_t + v_{xxx} + (u^n)_x v + u^n v_x = \dfrac{(u_x^n)^2}{2}, \\ v(0) = (u_0)_x. \end{cases} \tag{5.10.2}$$

By applying Lemma 5.6, we obtain that (5.10.2) admits a unique solution $v \in L_t^\infty H_x^1$. By Sobolev embedding, we obtain that $v \in L_{t,x}^\infty$, which implies that for almost every $t \in [0,T]$, $v(t)$ is locally integrable. Then we may define

$$u^{n+1}(t,x) = u_0(0) + \int_0^t \int_{-\infty}^0 \frac{(u_x^n(s,y))^2}{2} \, dy - v_{xx}(s,0) - u^n(s,0)v(s,0) \, ds + \int_0^x v(t,y) \, dy.$$

It is now straightforward to check that u^{n+1} formally solves (5.10.1). For the energy estimate, we apply the energy estimates of Lemmas 5.5 and 5.6 to $(u^{n+1})_x$ with the induction hypothesis to obtain that for every $t \in [0,T]$, with T chosen appropriately small depending on C and $\|u_0\|_X$ (the quantity $\|u_x^{n+1}\|_{L_x^\infty}$ can be controlled by $(E^{n+1})^{1/2} = \|u_x^{n+1}\|_{H_x^1}$ via Sobolev embedding),

$$\frac{d}{dt}(\|u_x^{n+1}\|_{H_x^1}^2) \lesssim \|(u_x^n)^2\|_{H_x^1}\|u_x^{n+1}\|_{H_x^1} + \|u_x^n\|_{L_x^\infty}\|u_x^{n+1}\|_{H_x^1}^2 + \|u_{xx}^n\|_{L_x^2}\|u_x^{n+1}\|_{L_x^\infty}\|u_x^{n+1}\|_{H_x^1},$$

hence

$$\frac{d}{dt}E^{n+1}(t) \lesssim E^n(t)(E^{n+1}(t))^{1/2} + (E^n(t))^{1/2}E^{n+1}(t).$$

By applying Grönwall's Lemma for $(E^{n+1}(t))^{1/2}$, we deduce that

$$\begin{aligned}(E^{n+1}(t))^{1/2} &\leq e^{\frac{C}{2}\int_0^t (E^n(s))^{1/2}\,ds}\left((E^{n+1}(0))^{1/2} + \frac{C}{2}\int_0^t e^{-\frac{C}{2}\int_0^s (E^n(\tau))^{1/2}\,d\tau}E^n(s)\,ds\right)\\ &\leq e^{\frac{CTE^{1/2}}{2}}\left(\frac{E^{1/2}}{2} + \frac{CTE}{2}\right) \lesssim E^{1/2}.\end{aligned}$$

In addition, the energy estimates for u^{n+1} show that it is a unique solution, hence the iteration is well-defined.

L_x^∞ control for u^{n+1}

By projecting (5.10.1) onto negative frequencies, we have

$$\begin{aligned}&(u_t^{n+1})_{\leq 0} + (u_{xxx}^{n+1})_{\leq 0} + (u^n u_x^{n+1})_{\leq 0} = \left(\frac{\partial_x^{-1}((u_x^n)^2)}{2}\right)_{\leq 0},\\ &(u^{n+1}(0))_{\leq 0} = (u_0)_{\leq 0}.\end{aligned}$$

Thus, for $t \in [0,T]$, we have (by Bernstein's inequalities and Sobolev embedding)

$$\begin{aligned}\|(u^{n+1})_{\leq 0}\|_{L_x^\infty} &\lesssim \|(u_0)_{\leq 0}\|_{L_x^\infty} + \int_0^t \|(u^{n+1}(s)_{xxx})_{\leq 0}\|_{L_x^\infty} + \|(u^n(s)u_x^{n+1}(s))_{\leq 0}\|_{L_x^\infty}\,ds\\ &\quad + \int_0^t \|\left(\partial_x^{-1}((u_x^n(s))^2)\right)_{\leq 0}\|_{L_x^\infty}\,ds\\ &\lesssim \|(u_0)_{\leq 0}\|_{L_x^\infty} + T\left(\|(u_{xxx}^{n+1})_{\leq 0}\|_{L_{t,x}^\infty} + \|(u^n u_x^{n+1})_{\leq 0}\|_{L_{t,x}^\infty} + \|u_x^n\|_{L_t^\infty L_x^2}^2\right)\\ &\lesssim \|u_0\|_{L_x^\infty} + T\left(\|(u_x^{n+1})_{\leq 0}\|_{L_{t,x}^\infty} + \|u^n\|_{L_{t,x}^\infty}\|u_x^{n+1}\|_{L_x^\infty} + \|u_x^n\|_{L_t^\infty L_x^2}^2\right)\\ &\lesssim \|u_0\|_{L_x^\infty} + T\left(\|u_x^{n+1}\|_{L_t^\infty H_x^1} + \|u^n\|_{L_{t,x}^\infty}\|u_x^{n+1}\|_{L_t^\infty H_x^1} + \|u_x^n\|_{L_t^\infty L_x^2}^2\right) < \infty\end{aligned}$$

Moreover, by Sobolev embedding for the high frequencies,

$$\|(u^{n+1})_{>0}\|_{L_{t,x}^\infty} \lesssim \|u^{n+1}\|_{\dot{H}_x^1\cap \dot{H}_x^2} < \infty,$$

hence $u^{n+1} \in L_{t,x}^\infty([0,T]\times\mathbb{R})$.

By applying Lemma 5.8, we have

$$\begin{aligned}\|(u^{n+1})_{\leq 0}\|_{L_{t,x}^\infty} &\lesssim \|(u_0)_{\leq 0}\|_{L_x^\infty} + T(\|(\partial_x^{-1}(u_x^n)^2)_{\leq 0}\|_{L_{t,x}^\infty} + \|u^n\|_{L_t^\infty W^{1,\infty}}\|u^{n+1}\|_{L_{t,x}^\infty})\\ &\lesssim \|(u_0)_{\leq 0}\|_{L_x^\infty} + T(\|u_x^n\|_{L_t^\infty L_x^2}^2 + \|u^n\|_{L_t^\infty X}\|u^{n+1}\|_{L_{t,x}^\infty})\\ &\lesssim \frac{1}{2}E^{1/2} + TE + TE^{1/2}\|u^{n+1}\|_{L_{t,x}^\infty}.\end{aligned}$$

Combined with Sobolev embedding for the high frequencies,

$$\|(u^{n+1})_{>0}\|_{L^\infty_{t,x}} \lesssim \|u^{n+1}\|_{\dot H^1_x \cap \dot H^2_x} \lesssim E^{1/2},$$

we obtain

$$\|u^{n+1}\|_{L^\infty_{t,x}} \lesssim E^{1/2} + TE + TE^{1/2}\|u^{n+1}\|_{L^\infty_{t,x}}.$$

By choosing T sufficiently small depending on E, we conclude that our iteration is well-defined with the uniform bound

$$\|u^{n+1}\|_{L^\infty_t X} \lesssim E^{1/2}.$$

5.10.2 Convergence for u^n

We shall now prove that u^n is a Cauchy sequence in $L^\infty_t(L^\infty_x \cap \dot H^1_x)$. Let $z = u^{n+2} - u^{n+1}$. In this case, z satisfies

$$z_t + u^{n+1}z_x + z_{xxx} = \frac{\partial_x^{-1}((u_x^{n+1})^2 - (u_x^n)^2)}{2} - (u^{n+1} - u^n)u_x^{n+1} =: H \tag{5.10.3}$$

and thus z_x satisfies

$$(z_x)_t + u_x^{n+1}z_x + u^{n+1}z_{xx} + z_{xxxx} = H_x. \tag{5.10.4}$$

We estimate the source term:

$$\begin{aligned}
\|H_x\|_{L^\infty_t L^2_x} &\le \|(u_x^{n+1})^2 - (u_x^n)^2\|_{L^\infty_t L^2_x} + \|(u^{n+1} - u^n)u_{xx}^{n+1}\|_{L^\infty_t L^2_x} + \|(u^{n+1} - u^n)_x u_x^{n+1}\|_{L^\infty_t L^2_x} \\
&\le \|u_x^{n+1} - u_x^n\|_{L^\infty_t L^2_x}\|u_x^{n+1} + u_x^n\|_{L^\infty_{t,x}} + \|u^{n+1} - u^n\|_{L^\infty_{t,x}}\|u_{xx}^{n+1}\|_{L^\infty_t L^2_x} \\
&\quad + \|u_x^{n+1} - u_x^n\|_{L^\infty_t L^2_x}\|u_x^{n+1}\|_{L^\infty_{t,x}} \\
&\lesssim E^{1/2}\|u^{n+1} - u^n\|_{L^\infty_t(L^\infty_x \cap \dot H^1_x)},
\end{aligned}$$

and

$$\begin{aligned}
\|H\|_{L^\infty_{t,x}} &\lesssim \|\partial_x^{-1}((u_x^{n+1})^2 - (u_x^n)^2)\|_{L^\infty_{t,x}} + \|(u^{n+1} - u^n)u_x^{n+1}\|_{L^\infty_{t,x}} \\
&\lesssim \|u_x^{n+1} - u_x^n\|_{L^\infty_t L^2_x}\|u_x^{n+1} + u_x^n\|_{L^\infty_t L^2_x} + \|u^{n+1} - u^n\|_{L^\infty_{t,x}}\|u_x^{n+1}\|_{L^\infty_{t,x}} \\
&\lesssim E^{1/2}\|u^{n+1} - u^n\|_{L^\infty_t(L^\infty_x \cap \dot H^1_x)}.
\end{aligned}$$

By applying the energy estimate provided by Lemma 5.5 and choosing T sufficiently small, we have

$$\begin{aligned}
\|z_x(t)\|_{L^2_x} &\le e^{\frac{C}{2}\int_0^t \|(u^{n+1}(s))_x\|_{L^\infty_x}\,ds}\left(\frac{C}{2}\int_0^t e^{-\frac{C}{2}\int_0^s \|(u^{n+1}(\tau))_x\|_{L^\infty_x}\,d\tau}\|H_x(s)\|_{L^2_x}\,ds\right) \\
&\lesssim e^{\frac{TCE^{1/2}}{2}} TE^{1/2}\|u^{n+1} - u^n\|_{L^\infty_t(L^\infty_x \cap \dot H^1_x)} \\
&\ll \|u^{n+1} - u^n\|_{L^\infty_t(L^\infty_x \cap \dot H^1_x)}.
\end{aligned}$$

For the L^∞ estimates, applying Lemma 5.8

$$\begin{aligned}\|z_{\le 0}\|_{L^\infty_{t,x}} &\lesssim T(\|H_{\le 0}\|_{L^\infty_{t,x}} + \|u^{n+1}\|_{L^\infty_t W^{1,\infty}_x}\|z\|_{L^\infty_t L^\infty_x} + \|z_{\le 0}\|_{L^\infty_{t,x}})\\ &\lesssim T(\|H\|_{L^\infty_{t,x}} + \|u^{n+1}\|_{L^\infty_t W^{1,\infty}_x}\|z\|_{L^\infty_t \dot H^1_x}) + T(1+\|u^{n+1}\|_{L^\infty_t W^{1,\infty}_x})\|z_{\le 0}\|_{L^\infty_{t,x}}\end{aligned}$$

and choosing T appropriately small depending on $(1+\|u^{n+1}\|_{L^\infty_t W^{1,\infty}_x}) \lesssim E$, we absorb into the left hand side to obtain

$$\begin{aligned}\|z_{\le 0}\|_{L^\infty_{t,x}} &\lesssim T(\|H_{\le 0}\|_{L^\infty_{t,x}} + \|u^{n+1}\|_{L^\infty_t W^{1,\infty}}\|z\|_{L^\infty_t \dot H^1_x})\\ &\lesssim TE^{1/2}(\|u^{n+1}-u^n\|_{L^\infty_t(L^\infty_x \cap \dot H^1_x)} + \|z\|_{L^\infty_t \dot H^1_x}).\end{aligned}$$

For the high frequencies, we use Sobolev embedding:

$$\|z_{>0}\|_{L^\infty_{t,x}} \lesssim \|z\|_{L^\infty_t \dot H^1_x} \ll \|u^{n+1}-u^n\|_{L^\infty_t(L^\infty_x\cap \dot H^1_x)}.$$

Putting everything together, and choosing T sufficiently small (depending on R), we get

$$\|u^{n+2}-u^{n+1}\|_{L^\infty_t(L^\infty_x\cap \dot H^1_x)} \le \frac{1}{2}\|u^{n+1}-u^n\|_{L^\infty_t(L^\infty_x\cap \dot H^1_x)}.$$

By iterating, we get

$$\|u^{n+2}-u^{n+1}\|_{L^\infty_t(L^\infty_x\cap \dot H^1_x)} \le 2^{-n-1}\|u^1-u^0\|_{L^\infty_t(L^\infty_x\cap \dot H^1_x)} \lesssim 2^{-n}E^{\frac{1}{2}},$$

which shows that u^n is a fundamental sequence in $L^\infty_t(\dot H^1_x \cap L^\infty_x)$ converging to an element $u \in L^\infty_t(\dot H^1_x \cap L^\infty_x)$. In particular, u^n_x converges to u_x in $L^\infty_t L^2_x$. As u^n_x is bounded in $L^\infty_t H^1_x$ (because u^n is bounded in $L^\infty_t X$), Lemma 5.9 implies that $u_x \in L^\infty_t H^1_x$. Therefore, $u \in L^\infty_t X$.

5.10.3 Uniqueness

Let u and v be two solutions to (5.1.1) with initial data $u(0) = u_0$ and $v(0) = v_0$ such that $\|u_0\|_X < R$ and $\|v_0\|_X < R$. Let $w = u - v$. Recall that we have the bounds $\|u\|_{L^\infty_t X}, \|v\|_{L^\infty_t X} \le E^{1/2}$.

In this case, w satisfies

$$w_t + uw_x + w_{xxx} = -wv_x + \frac{\partial_x^{-1}(w_x(u_x+v_x))}{2} =: H \tag{5.10.5}$$

so that w_x satisfies

$$(w_x)_t + uw_{xx} + \frac{(u_x+v_x)w_x}{2} + w_{xxxx} = -wv_{xx}. \tag{5.10.6}$$

By applying the energy estimate provided by Lemma 5.5 and choosing T sufficiently small, we get that

$$\begin{aligned} \|w_x\|_{L_t^\infty L_x^2} &\leq e^{\frac{C}{2}\int_0^t \|u_x(s)\|_{L_x^\infty} + \|v_x(s)\|_{L_x^\infty}\, ds}(\|(u_0)_x - (v_0)_x\|_{L_x^2} \\ &\quad + \frac{C}{2}\int_0^t e^{-\frac{C}{2}\int_0^s \|u_x(\tau)\|_{L_x^\infty} + \|v_x(\tau)\|_{L_x^\infty}\, d\tau}\|wv_{xx}\|_{L_x^2}\, ds) \\ &\lesssim \|(u_0)_x - (v_0)_x\|_{L_x^2} + TE^{1/2}\|w\|_{L_{t,x}^\infty}. \end{aligned} \tag{5.10.7}$$

For later use, we see that formally, we also have the energy estimate of Lemma 5.6,

$$\begin{aligned} \|w_x\|_{L_t^\infty H_x^1} &\leq e^{\frac{C}{2}\int_0^t \|u_x(s)\|_{L_x^\infty \cap \dot{H}_x^1} + \|v_x(s)\|_{L_x^\infty \cap \dot{H}_x^1}\, ds}(\|(u_0)_x - (v_0)_x\|_{H_x^1} \\ &\quad + \frac{C}{2}\int_0^t e^{-\frac{C}{2}\int_0^s \|u_x(\tau)\|_{L_x^\infty \cap \dot{H}_x^1} + \|v_x(\tau)\|_{L_x^\infty \cap \dot{H}_x^1}\, d\tau}\|wv_{xx}\|_{H_x^1}\, ds) \\ &\lesssim \|(u_0)_x - (v_0)_x\|_{H_x^1} + T\|v_{xx}\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}. \end{aligned} \tag{5.10.8}$$

For L^∞ estimates, we estimate the source term:

$$\begin{aligned} \|H\|_{L_x^\infty} = \left\| -wv_x + \frac{\partial_x^{-1}(w_x(u_x + v_x))}{2} \right\|_{L_x^\infty} &\lesssim \|w_x\|_{L_t^\infty L_x^2}\|u_x + v_x\|_{L_t^\infty L_x^2} + \|w\|_{L_{t,x}^\infty}\|v_x\|_{L_{t,x}^\infty} \\ &\lesssim \|(u_x, v_x)\|_{L_t^\infty(L_x^2 \cap L_x^\infty)}\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \\ &\lesssim \|(u_x, v_x)\|_{L_t^\infty H_x^{\frac{1}{2}+}}\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}. \end{aligned}$$

Then applying Lemma 5.8 and choosing T appropriately small, we have

$$\begin{aligned} \|w_{\leq 0}\|_{L_{t,x}^\infty} &\lesssim \|(w(0))_{\leq 0}\|_{L_x^\infty} + T(\|H_{\leq 0}\|_{L_{t,x}^\infty} + \|u\|_{L_t^\infty W^{1,\infty}}\|w\|_{L_t^\infty \dot{H}_x^1}) \\ &\lesssim \|u_0 - v_0\|_{L_x^\infty} + T(\|H\|_{L_{t,x}^\infty} + \|u\|_{L_t^\infty W^{1,\infty}}\|w\|_{L_t^\infty \dot{H}_x^1}) \\ &\lesssim \|u_0 - v_0\|_{L_x^\infty} + T\|(u_x, v_x)\|_{L_t^\infty H^{\frac{1}{2}+}}\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \\ &\lesssim \|u_0 - v_0\|_{L_x^\infty} + TE^{1/2}\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \end{aligned} \tag{5.10.9}$$

Moreover, by Sobolev embedding,

$$\|w_{>0}\|_{L_{t,x}^\infty} \lesssim \|w_x\|_{L_t^\infty L_x^2}.$$

By adding this inequality, along with equations 5.10.7 and 5.10.9, we get that

$$\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \lesssim \|(u_0)_x - (v_0)_x\|_{L_x^2} + \|u_0 - v_0\|_{L_x^\infty} + TE^{1/2}\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}.$$

Choosing T sufficiently small, we find

$$\|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \lesssim \|u_0 - v_0\|_{L_x^\infty \cap \dot{H}_x^1} \tag{5.10.10}$$

which establishes uniqueness.

5.10.4 Continuity with respect to the initial data

Consider a sequence of initial data

$$u_{0j} \to u_0 \in X.$$

Here, since $\|u_0\|_X < R$, we may assume that $\|u_{0j}\|_X < R$ for every j, and the existence part implies that u_j and u may be defined on a common time interval $[0, T]$, with uniform bounds in j. Furthermore, by the Lipschitz estimate from the proof of uniqueness,

$$u_j \to u \in L_t^\infty(L_x^\infty \cap \dot{H}_x^1).$$

By interpolation, it follows that

$$u_j \to u \in L_t^\infty([0, T], L_x^\infty \cap \dot{H}_x^1 \cap \dot{H}_x^{2-\varepsilon}).$$

To obtain the endpoint, we take an approach similar to the one presented in [87].

We define $u_{0j}^h = (u_{0j})_{\leq h}$ and $u_0^h = (u_0)_{\leq h}$, and may assume that

$$\|u_{0j}^h\|_X \lesssim \|u_0\|_X,$$

so that there exist $T = T(\|u_0\|_X) > 0$ and solutions u^h and u_j^h that belong to $L_t^\infty X$. Further, Lemma 5.7 shows that u^h and u_j^h belong to $L_t^\infty(X \cap \dot{H}_x^3)$. As (by Sobolev embedding)

$$\int_0^T \|u_x^h(s)\|_{L_x^\infty}\, ds \lesssim T\|u^h\|_{L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^2)},$$

we have from the energy estimate of Lemma 5.7 that

$$\|u^h\|_{L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^3)} \lesssim \|u_0^h\|_{\dot{H}_x^1 \cap \dot{H}_x^3},$$

and likewise for u_j^h.

We consider H_x^1 sharp frequency envelopes for $(u_0)_x$ and $(u_{0j})_x$, denoted by $\{c_k\}_{k\in\mathbb{Z}}$ and $\{c_k^j\}_{k\in\mathbb{Z}}$. As $(u_{0j})_x \to (u_0)_x$ in H_x^1, we can assume that $c_k^j \to c_k$ in l^2. Moreover, as in [87], we can choose c_k having the following properties:

a) Uniform bounds:
$$\|P_k(u_0^h)_x\|_{H_x^1} \lesssim c_k$$

b) High frequency bounds:
$$\|(u_0^h)_x\|_{H_x^2} \lesssim 2^h c_h$$

c) Difference bounds:
$$\|u_0^{h+1} - u_0^h\|_{\dot{H}_x^1} \lesssim 2^{-h} c_h$$

d) Limit as $h \to \infty$:

$$D_x u_0^h \to D_x u_0 \in H_x^1$$

and likewise for c_k^j.

We first establish estimates for $(u - u^h)_{>0}$ and $(u_j - u_j^h)_{>0}$ in $L_t^\infty X$. We treat the low frequencies separately because the frequency envelopes that we are using are $\dot{H}^1 \cap \dot{H}^2$-based, and don't allow us to control the L^∞-component of the norm of X at low frequencies. By applying the Lipschitz estimate from the proof of uniqueness, we can see that (for $h > 0$)

$$\|u^{h+1} - u^h\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^\infty)} \lesssim \|u_0^{h+1} - u_0^h\|_{\dot{H}_x^1 \cap L_x^\infty} \lesssim \|u_0^{h+1} - u_0^h\|_{\dot{H}_x^1} \lesssim 2^{-h} c_h.$$

Taking the high frequencies and interpolating with the estimate

$$\|u_{>0}^h\|_{L_t^\infty(X \cap \dot{H}_x^3)} \lesssim \|u^h\|_{L_t^\infty(\dot{H}_x^1 \cap \dot{H}_x^3)} \lesssim \|u_0^h\|_{\dot{H}_x^1 \cap \dot{H}_x^3} \lesssim 2^h c_h,$$

we get that

$$\|u_{>0}^{h+1} - u_{>0}^h\|_{L_t^\infty X} \lesssim c_h.$$

The analogous analysis and estimates hold for u_j^h. Moreover, as in [87], we get that

$$\|u_{>0} - u_{>0}^h\|_{L_t^\infty X} \lesssim c_{\geq h} = \left(\sum_{k \geq h} c_k^2 \right)^{1/2}, \quad \|(u_j)_{>0} - (u_j^h)_{>0}\|_{L_t^\infty X} \lesssim c_{\geq h}^j = \left(\sum_{k \geq h} (c_k^j)^2 \right)^{1/2}.$$

Next, we show that for fixed h, $\lim_{j \to \infty} u_j^h = u^h$ in $L_t^\infty X([0, T] \times \mathbb{R})$. Observe that by (5.10.8), $w = u^h - u_j^h$ satisfies

$$\|w_x\|_{L_t^\infty H_x^1} \lesssim \|w_x(0)\|_{H_x^1} + T \|(u_j^h)_{xx}\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \|w\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}$$

or equivalently

$$\|(u^h - u_j^h)_x\|_{L_t^\infty H_x^1} \lesssim \|u_{0j} - u_{0j}^h\|_{H_x^1} + T \|(u_j^h)_{xx}\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)} \|u^h - u_j^h\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}.$$

As h is fixed, the previous discussion ensures that $\|(u_j^h)_{xx}\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^1)}$ is uniformly bounded with respect to j. Then (5.10.10) implies that $\lim_{j \to \infty} u_j^h = u^h$ in $L_t^\infty(L_x^\infty \cap \dot{H}_x^1)$, which along with $\lim_{j \to \infty} u_{0j}^h = u_0^h$ in X implies that $\lim_{j \to \infty} u_j^h = u^h$ in $L_t^\infty \dot{H}_x^2$. Thus, $\lim_{j \to \infty} u_j^h = u^h$ in $L_t^\infty X$, as claimed.

To complete the argument, we have

$$\begin{aligned} \|u_{>0} - (u_j)_{>0}\|_{L_t^\infty X} &\lesssim \|u^h - u_j^h\|_{L_t^\infty X} + \|u_{>0} - u_{>0}^h\|_{L_t^\infty X} + \|(u_j)_{>0} - (u_j^h)_{>0}\|_{L_t^\infty X} \\ &\lesssim \|u^h - u_j^h\|_{L_t^\infty X} + c_{\geq h} + c_{\geq h}^j \end{aligned}$$

so that fixing h,

$$\limsup_{j\to\infty} \|u_{>0} - (u_j)_{>0}\|_{L_t^\infty X} \lesssim c_{\geq h} + c^j_{\leq h}$$

Then letting h tend to ∞, we get that

$$\lim_{j\to\infty} \|u_{>0} - (u_j)_{>0}\|_{L_t^\infty X} = 0.$$

For the low frequencies, we directly estimate (using the Lipschitz bound from the proof of uniqueness)

$$\|u_{\leq 0} - (u_j)_{\leq 0}\|_{L_t^\infty X} \lesssim \|u_{\leq 0} - (u_j)_{\leq 0}\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^\infty)} \lesssim \|u - u_j\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^\infty)} \lesssim \|u_0 - u_{0j}\|_{\dot{H}_x^1 \cap L_x^\infty}$$

As $u_{j0} \to u_j$ in X, it follows that

$$\lim_{j\to\infty} \|u_{\leq 0} - (u_j)_{\leq 0}\|_{L_t^\infty X} = 0.$$

Combining the low and high frequencies, we obtain $u_j \to u$ in $L_t^\infty X$.

5.10.5 Continuity in time

Let $h > 0$ be an arbitrary parameter, and u^h solve (5.1.1) with initial data $(u_0)_{\leq h}$. In particular,

$$u_t^h = \frac{\partial_x^{-1}((u_x^h)^2)}{2} - u^h u_x^h - u_{xxx}^h. \tag{5.10.11}$$

From Lemma 5.7, we know that $u^h \in L_t^\infty(X \cap \dot{H}_x^5)$, so that the right hand side belongs to $L_t^\infty X$. Thus, $u^h \in C_t^0 X$. From the previous section, we know that u^h converges to u in $L_t^\infty X$, hence in $C_t^0 X$. This concludes the proof of Theorem 5.1.1.

5.11 Global well-posedness

In this section, we prove Theorem 5.1.2. Recall that the dispersive Hunter-Saxton (5.1.1) has the conserved quantities (see [52])

$$E_1(t) = \int_{\mathbb{R}} u_x(t)^2 \, dx$$
$$E_2(t) = \int_{\mathbb{R}} u_{xx}(t)^2 - u(t)u_x(t)^2 \, dx.$$

Throughout the proof, $C > 0$ shall denote a universal large constant. Consider a solution u of (5.1.1) on $[0, T)$ where T is finite. We shall determine a uniform bound for $\|u(t)\|_X$.

We begin with the L^∞ estimate. The high frequencies can be controlled by the $\dot{H}^1$ norm, which is conserved via E_1, but the low frequencies need to be treated separately as follows. Projecting (5.1.1) onto frequencies less than or equal to 1, we consider

$$(u_{\leq 0})_t + (uu_x)_{\leq 0} + (u_{\leq 0})_{xxx} = \frac{(\partial_x^{-1}(u_x^2))_{\leq 0}}{2}.$$

For the transport term, write

$$\begin{aligned}(uu_x)_{\leq 0} - u_{\leq 0}(u_{\leq 0})_x &= (u_{>0}u_x)_{\leq 0} + [P_{\leq 0}, u_{\leq 0}]u_x \\ &= (u_{>0}u_x)_{\leq 0} + [P_{\leq 0}, P_0 u]u_x + [P_{\leq 0}, u_{<0}]P_0 u_x\end{aligned}$$

and estimate, using Sobolev embedding,

$$\|(u_{>0}u_x)_{\leq 0}\|_{L^\infty_x} \lesssim \|u_{>0}u_x\|_{L^2_x} \lesssim \|u_{>0}\|_{L^\infty_x}\|u_x\|_{L^2_x} \lesssim \|u_{>0}\|_{H^1_x}\|u_x\|_{L^2_x} \lesssim \|u_x\|^2_{L^2_x}.$$

The same estimate holds for the first commutator directly, without using the commutator structure. For the second commutator, by using Bernstein's inequalities with frequency localization $P_{\leq 0}$, followed by the Kato-Ponce commutator estimate Theorem 5.2.3, we get that

$$\|[P_{\leq 0}, u_{<0}]P_0 u_x\|_{L^\infty} \lesssim \|[P_{\leq 0}, u_{<0}]P_0 u_x\|_{L^2} \lesssim \|\partial_x u_{<0}\|_{L^\infty}\|P_0 u\|_{L^2} \lesssim \|u_x\|^2_{L^2}.$$

Besides this, we may estimate the dispersive and source terms by

$$\begin{aligned}\|(u_{\leq 0})_{xxx}\|_{L^\infty_x} &\lesssim \|(u_{\leq 0})_x\|_{L^\infty_x} \lesssim \|u_x\|_{L^2_x} \lesssim \|u_x\|_{L^2_x}, \\ \|(\partial_x^{-1}(u_x^2))_{\leq 0}\|_{L^\infty_x} &\lesssim \|\partial_x^{-1}(u_x^2)\|_{L^\infty_x} \lesssim \|u_x\|^2_{L^2_x},\end{aligned}$$

where we have used Bernstein's inequalities for the dispersive term $(u_{\leq 0})_{xxx}$. Therefore, denoting

$$F = \frac{(\partial_x^{-1}(u_x^2))_{\leq 0}}{2} - ((uu_x)_{\leq 0} - u_{\leq 0}(u_{\leq 0})_x) - (u_{\leq 0})_{xxx},$$

we have

$$\|F\|_{L^\infty_x} \lesssim \|u_x\|^2_{L^2_x} + \|u_x\|_{L^2_x} = E_1 + E_1^{1/2}.$$

As $u \in C^0_t X([0,T)\times\mathbb{R})$, we see that u is continuous with respect to t and x, and Lipschitz with respect to x, uniformly in t. As in [144], let us consider the flow

$$q_t = u_{\leq 0}(t, q(t,x)), \qquad q(0,x) = x.$$

By standard ordinary differential equations theory, q exists, is unique, and is defined on the whole interval $[0,T)$ as a function in $C^1([0,T))$. Moreover, it is not difficult to see that it is a C^1-diffeomorphism. We also note that $q_{xt} = u_x q_x$, which means that $q_x = e^{\int_0^t u_x(s,q(s,x))\,ds} > 0$, hence q is strictly increasing in x for every t. Further,

$$\frac{d}{dt}u_{\leq 0}(t, q(t,x)) = (u_{\leq 0})_t + u_{\leq 0}(u_{\leq 0})_x = F.$$

Then

$$\|u_{\leq 0}(t,q(t,x))\|_{L^\infty_x} \lesssim \|u_{\leq 0}(0)\|_{L^\infty_x} + \int_0^t \|F\|_{L^\infty_x}\,ds \lesssim \|(u_0)_{\leq 0}\|_{L^\infty_x} + \int_0^t E_1 + E_1^{1/2}\,ds.$$

As q is a diffeomorphism, we now infer that

$$\|(u(t))_{\leq 0}\|_{L^\infty_x} \lesssim \|(u_0)_{\leq 0}\|_{L^\infty_x} + t(E_1 + E_1^{1/2}).$$

For the high frequencies, we apply Sobolev embeddings and Bernstein's inequalities to estimate

$$\|(u(t))_{>0}\|_{L^\infty_x} \lesssim E_1^{1/2}.$$

Combining these estimates, we conclude that for every $t \in [0,T)$,

$$\|u(t)\|_{L^\infty_x} \lesssim \|u_0\|_{X^0} + t(E_1 + E_1^{1/2}).$$

Thus, for some constant $C > 0$, and for every $t \in [0,T)$, we have

$$\begin{aligned}\|u_{xx}(t)\|^2_{L^2_x} &\lesssim |E_2| + \|u(t)\|_{L^\infty_x}\|u_x(t)\|^2_{L^2_x}\\ &\lesssim \|u_0\|^2_{\dot{H}^2_x} + \|u_0\|_{X^0}E_1 + t(E_1 + E_1^{1/2})E_1.\end{aligned}$$

We obtain the desired estimate for $\|u(t)\|_X$, where $t \in [0,T)$. In particular, the lifespan for u may be extended indefinitely.

5.12 A normal form analysis

In this section, we use normal forms to construct an energy functional corresponding to $\dot{H}^{1+s}$. Since (5.1.1) exhibits a quasilinear behavior at low frequencies, we use a modified energy approach as introduced in [72].

We may re-express the dispersive Hunter-Saxton (5.1.1) as

$$u_t + u_{xxx} = \partial_x^{-2}(u_x u_{xx}) - uu_x =: Q_1 + Q_2 =: Q. \tag{5.12.1}$$

The appropriate normal form variable $\tilde{u}$, which should formally satisfy an equation with cubic or higher order nonlinearities, may be computed directly for this Q. However, we observe that instead, the transformation may be formally deduced from the normal form transformation for the KdV equation (see for instance Section 5 in [82]), closely related to the well-known Miura transform (see the discussion in Chapter 4 of [137]). In the KdV setting,

$$u_t + u_{xxx} = 6uu_x = 6Q_2,$$

we have the formal normal form variable

$$\tilde{u} = u - (\partial_x^{-1} u)^2.$$

Thus, we see that the appropriate normal form transformation for (5.1.1) is

$$\tilde{u} = u + B(u, u) = u - \frac{1}{6}\partial_x^{-2}(u^2) + \frac{1}{6}(\partial_x^{-1} u)^2.$$

To construct a modified energy for $\dot{H}^{1+s}$, write

$$A(D) = D^s P_{>0}$$

and consider

$$\int Au_x \cdot A\left(u_x - \frac{1}{3}\partial_x^{-1}(u^2) + \frac{2}{3}(\partial_x^{-1} u)u\right)\, dx,$$

which is obtained by retaining only the cubic and lower order terms of

$$\int A\tilde{u}_x \cdot A\tilde{u}_x \, dx.$$

Integrating by parts on the last two terms and rearranging, we obtain

$$\int (Au_x)^2 - \frac{1}{3}Au \cdot A\left(u^2 + 2\partial_x^{-1} u \cdot u_x\right)\, dx.$$

Then commuting A through the last term, we have

$$\int (Au_x)^2 - \frac{1}{3}Au \cdot (A(u^2) + 2[A, \partial_x^{-1} u]u_x + 2\partial_x^{-1} u \cdot Au_x)\, dx.$$

Lastly, integrating by parts on the last term, we define the modified energy

$$\tilde{E}(t) := \int (Au_x)^2 - \frac{1}{3}Au \cdot (A(u^2) + 2[A, \partial_x^{-1} u]u_x - Au \cdot u)\, dx.$$

Lemma 5.13. If $u \in C_t^0 X^s([0, T) \times \mathbb{R})$, then for every $t \in [0, T)$, we have

$$\|(u(t))_{>0}\|^2_{\dot{H}_x^{1+s}} = \tilde{E}(t) + O(E_1 \|u(t)\|_{L_x^\infty}),$$

and

$$\frac{d}{dt}\tilde{E}(t) \lesssim \|Au\|^2_{L_x^2}(\|u_x\|^2_{L_x^2} + \|u_x\|_{L_x^\infty}\|u\|_{L_x^\infty}).$$

Proof. We have

$$\begin{aligned} \|[A, \partial_x^{-1} v]w_x\|_{L_x^2} &\lesssim \|Aw\|_{L_x^2}\|v\|_{L_x^\infty} + \|Av\|_{L_x^2}\|w\|_{L_x^\infty}, \\ \|A(vw)\|_{L_x^2} &\lesssim \|Av\|_{L_x^2}\|w\|_{L_x^\infty} + \|Aw\|_{L_x^2}\|v\|_{L_x^\infty}. \end{aligned}$$

Thus, the first bound is immediate.

We now prove the energy estimate. First observe that $\frac{d}{dt}\tilde{E}$ consists only of quartic terms. Precisely, if we set

$$L_A(v,w) := -\frac{1}{3}A(vw) - \frac{2}{3}[A, \partial_x^{-1}v]w_x + \frac{1}{3}Av \cdot w,$$

so that

$$\tilde{E} = \int (Au_x)^2 - Au \cdot L_A(u,u)\,dx,$$

then a straightforward computation shows that

$$\frac{d}{dt}\tilde{E} = \int AQ \cdot L_A(u,u) + Au \cdot L_A(Q,u) + Au \cdot L_A(u,Q)\,dx,$$

where recall that $Q = Q_1 + Q_2$ is defined in (5.12.1).

We consider first the contribution from Q_1. Since

$$\|L_A(v,w)\|_{L^2_x} \lesssim \|Av\|_{L^2_x}\|w\|_{L^\infty_x} + \|Aw\|_{L^2_x}\|v\|_{L^\infty_x},$$

we have

$$\begin{aligned}\int AQ_1 \cdot L_A(u,u) + Au \cdot L_A(Q_1,u) + Au \cdot L_A(u,Q_1)\,dx& \\ \lesssim \|Au\|_{L^2_x}(\|AQ_1\|_{L^2_x}\|u\|_{L^\infty_x} + \|Au\|_{L^2_x}\|Q_1\|_{L^\infty_x}).&\end{aligned}$$

To bound Q_1, we have

$$\|\partial_x^{-1}(u_x^2)\|_{L^\infty_x} \lesssim \|u_x\|^2_{L^2_x}$$

and

$$\begin{aligned}\|A\partial_x^{-1}(u_x^2)\|_{L^2_x} &\lesssim \|A\partial_x^{-1}(T_{u_x}u_x)\|_{L^2_x} + \|A\partial_x^{-1}\Pi(u_x,u_x)\|_{L^2_x} \\ &\lesssim \|u_x\|_{L^\infty_x}\|Au\|_{L^2_x} + \|A\partial_x^{-1}\Pi(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}u_x)\|_{L^2_x} \\ &\lesssim \|u_x\|_{L^\infty_x}\|Au\|_{L^2_x} + \|A\Pi(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}u)\|_{L^2_x} \\ &\lesssim \|u_x\|_{L^\infty_x}\|Au\|_{L^2_x} + \|u_x\|_{BMO}\|Au\|_{L^2_x} \lesssim \|u_x\|_{L^\infty_x}\|Au\|_{L^2_x}.\end{aligned}$$

which suffice.

For the contribution from Q_2, we consider each of the three terms in

$$L_A(u,u) = -\frac{1}{3}A(u^2) - \frac{2}{3}[A, \partial_x^{-1}u]u_x + \frac{1}{3}Au \cdot u$$

successively. From the third term, and the Q_2 contribution arising from the case where the time derivative falls on the lone u,

$$\int \frac{1}{3} Au \cdot Au \cdot uu_x \, dx \lesssim \|u\|_{L^\infty} \|u_x\|_{L^\infty} \|Au\|_{L^2}^2.$$

On the other hand, when the derivative falls on Au, we write

$$\int \frac{1}{6} Au \cdot A\partial_x(u^2) \cdot u \, dx = \int \frac{1}{3} Au \cdot [A, u]u_x \cdot u + \frac{1}{3} Au \cdot u \cdot Au_x \cdot u \, dx.$$

The latter term is the same as the previous case after an integration by parts, while

$$\int \frac{1}{3} Au \cdot [A, u]u_x \cdot u \, dx \lesssim \|Au\|_{L_x^2} \|u\|_{L^\infty} \|[A, u]u_x\|_{L_x^2} \lesssim \|Au\|_{L_x^2}^2 \|u_x\|_{L_x^\infty} \|u\|_{L_x^\infty}.$$

From the first term in L_A, the case when the time derivative falls on Au vanishes via an integration by parts. Then from the remaining contribution,

$$\int \frac{1}{3} Au \cdot A\partial_x(u^3) \, dx = \int Au \cdot [A, u^2]u_x + Au \cdot u^2 \cdot Au_x \, dx.$$

The latter term has already appeared, while

$$\int Au \cdot [A, u^2]u_x \, dx \lesssim \|Au\|_{L_x^2} \|[A, u^2]u_x\|_{L_x^2} \lesssim \|Au\|_{L_x^2}^2 \|(u^2)_x\|_{L_x^\infty} \lesssim \|Au\|_{L_x^2}^2 \|u_x\|_{L_x^\infty} \|u\|_{L_x^\infty}.$$

Lastly, we have the commutator term from L. When the time derivative falls inside the commutator, we have

$$\int Au \cdot [A, \partial_x^{-1}(uu_x)]u_x \, dx \lesssim \|Au\|_{L_x^2} \|[A, \partial_x^{-1}(uu_x)]u_x\|_{L_x^2} \lesssim \|Au\|_{L_x^2}^2 \|u_x\|_{L_x^\infty} \|u\|_{L_x^\infty}.$$

From the remaining contributions of Q_2, we are left with

$$\int A(uu_x) \cdot [A, \partial_x^{-1}u]u_x + Au \cdot [A, \partial_x^{-1}u](uu_x)_x \, dx.$$

Integrating by parts on the second term, and since

$$\int Au \cdot [A, u](uu_x) \, dx \lesssim \|Au\|_{L_x^2} \|u_x\|_{L_x^\infty} \|A(u^2)\|_{L_x^2} \lesssim \|Au\|_{L_x^2}^2 \|u_x\|_{L_x^\infty} \|u\|_{L_x^\infty},$$

it remains to bound

$$\int A(uu_x) \cdot [A, \partial_x^{-1}u]u_x - Au_x \cdot [A, \partial_x^{-1}u](uu_x) \, dx = -\int u_x \cdot [A[A, \partial_x^{-1}u], u]u_x \, dx. \quad (5.13.1)$$

Before exploiting the full commutator structure, we first reduce to paraproducts.

From the first integral on the left hand side of (5.13.1), we write

$$\begin{aligned}\int A(uu_x)\cdot[A,\partial_x^{-1}u]u_x\,dx &= \int A(uu_x)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx\\ &\quad-\frac{1}{2}\int A(u^2)\cdot\partial_x(A(T_{u_x}\partial_x^{-1}u)+A\Pi(u_x,\partial_x^{-1}u))\,dx\\ &\quad+\frac{1}{2}\int A(u^2)\cdot\partial_x(T_{Au_x}\partial_x^{-1}u+\Pi(Au_x,\partial_x^{-1}u))\,dx.\end{aligned}$$

The last two lines are perturbative and may be discarded. Precisely, we have

$$\|A(u^2)\|_{L_x^2}\lesssim\|u\|_{L_x^\infty}\|Au\|_{L_x^2}$$

while

$$\begin{aligned}\|\partial_x A(T_{u_x}\partial_x^{-1}u)\|_{L_x^2}&\lesssim\|u_x\|_{L_x^\infty}\|Au\|_{L_x^2},\\ \|\partial_x(T_{Au_x}\partial_x^{-1}u)\|_{L_x^2}&\lesssim\|u_x\|_{L_x^\infty}\|Au\|_{L_x^2},\end{aligned}$$

with the same estimate for the balanced frequency terms.

Next, we proceed further to write

$$\begin{aligned}\int A(uu_x)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx &= \int A(T_uu_x)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx\\ &\quad+\int A(T_{u_x}u)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx+\int A\Pi(u_x,u)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx.\end{aligned}$$

The second line is perturbative as before. Precisely,

$$\int A(T_{u_x}u)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx\lesssim\|u_x\|_{L_x^\infty}\|Au\|_{L_x^2}\cdot\|u\|_{L_x^\infty}\|Au\|_{L_x^2}$$

with the same estimate for the balanced frequency term.

A similar analysis holds for the second term on the left hand side of (5.13.1), so we are only left to estimate

$$\int A(T_uu_x)\cdot[A,T_{\partial_x^{-1}u}]u_x\,dx-\int Au_x\cdot[A,T_{\partial_x^{-1}u}](T_uu_x)\,dx=-\int u_x\cdot[A[A,T_{\partial_x^{-1}u}],T_u]u_x\,dx.$$

Define

$$L(u,v,w)=D^{-s}\partial_x[A[A,T_{\partial_x^{-1}u}],T_{\partial_x^{-1}v}]D^{-s}w_x$$

and let L_k denote the frequency k component.

Let $a(\xi)=|\xi|^s(1-\phi(\xi))$ be the symbol of A, where ϕ is the symbol of the Littlewood-Paley projector $P_{\leq 0}$, and

$$\phi_k(\xi)=\phi\left(\frac{\xi}{2^{k-4}}\right),\qquad \psi_k(\xi)=\phi\left(\frac{\xi}{2^k}\right)-\phi\left(\frac{\xi}{2^{k-1}}\right).$$

The symbol of L_k is

$$L_k(\xi,\eta,\zeta) = \phi_k(\xi)\phi_k(\eta)\zeta\psi_k(\zeta)(\xi\eta|\xi+\eta+\zeta|^s|\zeta|^s)^{-1} \\ \cdot (a(\xi+\eta+\zeta)(a(\xi+\eta+\zeta)-a(\eta+\zeta)) - a(\xi+\zeta)(a(\xi+\zeta)-a(\zeta)))\,.$$

This symbol is supported in the region $\{(\xi,\eta,\zeta)|\xi,\eta \lesssim 2^k, \zeta \sim 2^k\}$, is smooth, and its associated kernel is bounded and integrable. Thus, as in [136] and [85] (see the the paragraph regarding multilinear forms in the second section of each paper), we have the following estimate for the multilinear form L:

$$-\int (u_k)_x \cdot [A[A,\partial_x^{-1}u_{<k}],u_{<k}](u_k)_x\,dx = \int Au_k \cdot L_k(u,u_x,Au_k)\,dx \\ \lesssim \|u\|_{L_x^\infty}\|u_x\|_{L_x^\infty}\|Au_k\|_{L_x^2}^2.$$

Thus,

$$\int u_x \cdot [A[A,T_{\partial_x^{-1}u}],T_u]u_x\,dx \lesssim \sum_k \|u\|_{L_x^\infty}\|u_x\|_{L_x^\infty}\|Au_k\|_{L_x^2}^2 \lesssim \|u\|_{L_x^\infty}\|u_x\|_{L_x^\infty}\|Au\|_{L_x^2}^2$$

By putting everything together, we obtain the desired estimate.

□

By combining the previous result with the L_x^∞ bounds from Theorem 5.1.2, we establish bounds on the growth of the solutions in $\dot{H}^{1+s}$.

Corollary 5.14. Let $T>0$, $I=[0,T]$ or $I=[0,T)$, and $u \in C_t^0X^s(I\times\mathbb{R})$ solve (5.1.1). Then we have the bounds (5.1.5).

Proof. We have from Theorem 5.1.2 the pointwise estimates. It remains to establish the energy bounds.

Let $\tilde{E}$ be the modified energy functional of Lemma 5.13, so that for $t\in[0,T)$,

$$\frac{d}{dt}\tilde{E}(t) \lesssim \|Au\|_{L_x^2}^2(\|u_x\|_{L_x^2}^2 + \|u_x\|_{L_x^\infty}\|u\|_{L_x^\infty}) \\ \lesssim \|u_x(t)\|_{L_x^2}^4 + \|u_x(t)\|_{L_x^2}^3\|u(t)\|_{L_x^\infty} + \|u_x(t)\|_{L_x^2}^2\|u(t)\|_{L_x^\infty}\|u(t)_{>0}\|_{\dot{H}_x^{1+s}}$$

and since we have the energy equivalence

$$\|(u(t))_{>0}\|_{\dot{H}_x^{1+s}}^2 = \tilde{E}(t) + O(E_1\|u(t)\|_{L_x^\infty}),$$

we find

$$\frac{d}{dt}\tilde{E}(t) \lesssim E_1^2 + E_1^{3/2}\|u(t)\|_{L_x^\infty} + E_1\|u(t)\|_{L_x^\infty}(\tilde{E}(t)+CE_1\|u(t)\|_{L_x^\infty})^{1/2} \\ \lesssim E_1^2 + E_1\|u(t)\|_{L_x^\infty}(\tilde{E}(t)+E_1+CE_1\|u(t)\|_{L_x^\infty})^{1/2} \\ \lesssim E_1\|u(t)\|_{X^0}(\tilde{E}(t)+E_1+CE_1\|u(t)\|_{L_x^\infty})^{1/2} \\ \lesssim E_1\|u\|_{L_t^\infty X^0}(\tilde{E}(t)+E_1+CE_1\|u\|_{L_{t,x}^\infty})^{1/2}.$$

Integrating in t, we find that for every $t \in [0, T)$,

$$\left(CE_1\|u\|_{L^\infty_{t,x}} + E_1 + \tilde{E}(t)\right)^{\frac{1}{2}} \lesssim tE_1\|u\|_{L^\infty_t X^0} + \left(CE_1\|u\|_{L^\infty_{t,x}} + E_1 + \tilde{E}(0)\right)^{\frac{1}{2}}.$$

Thus,

$$\tilde{E}(t) \lesssim t^2 E_1^2 \|u\|^2_{L^\infty_t X^0} + E_1(\|u\|_{L^\infty_{t,x}} + 1) + \tilde{E}(0).$$

Using the first inequality from Lemma 5.13 and the low frequency bound

$$\|(u(t))_{\leq 0}\|^2_{\dot{H}^{1+s}_x} \lesssim E_1,$$

we have

$$\|u(t)\|^2_{\dot{H}^{1+s}_x} \lesssim t^2 E_1^2 \|u\|^2_{L^\infty_t X^0} + E_1(\|u\|_{L^\infty_{t,x}} + 1) + \|u_0\|^2_{\dot{H}^{1+s}_x}.$$

Combined with the pointwise estimates, we obtain the stated bound. □

This establishes the bounds in Theorem 5.1.4. Global well-posedness now follows from the local result of Theorem 5.1.3, which we prove in the next two sections.

5.15 An estimate for the linearized equation

The linearized equation corresponding to (5.1.1) is

$$w_t + (uw)_x + w_{xxx} = \partial_x^{-1}(u_x w_x),$$

which can be rewritten as

$$w_t + \partial_x^{-1}(u_{xx} w) + u w_x + w_{xxx} = f. \tag{5.15.1}$$

Applying D^s with $s \in (\frac{1}{2}, 1)$ to (5.15.1), and writing $v = D_x^s w$, we have

$$v_t + u v_x + v_{xxx} = -D_x^s \partial_x^{-1}(T_{u_{xx}} w + T_w u_{xx} + \Pi(w, u_{xx})) - [D_x^s, u] w_x + D_x^s f. \tag{5.15.2}$$

Lemma 5.16. Let $T > 0$ and $I = [0, T]$. If $u \in L^\infty_t X^s$ is a solution of (5.1.1) in I and $w \in C^0_t(L^\infty_x \cap \dot{H}^s_x)(I \times \mathbb{R})$ is a solution of (5.15.1), then by shrinking T enough depending on $\|u\|_{L^\infty_t X^s(I\times\mathbb{R})}$, we have

$$\|w\|_{L^\infty_t(L^\infty_x \cap \dot{H}^s_x)} \lesssim \|w_0\|_{L^\infty_x \cap \dot{H}^s_x} + \|f\|_{L^1_t(L^\infty_x \cap \dot{H}^s_x)}.$$

Proof. We consider the homogeneous problem with $f = 0$, as the proof below easily generalizes. We first bound the source terms of (5.15.2) in L^2. For the first two source terms, we have

$$\|D_x^s \partial_x^{-1}(T_{u_{xx}} w)\|_{L^2_x} \lesssim \|w\|_{\dot{H}^s_x} \|u_x\|_{L^\infty_x} \lesssim \|w\|_{\dot{H}^s_x} \|u\|_{\dot{H}^1_x \cap \dot{H}^{1+s}_x}$$

and

$$\|D_x^s \partial_x^{-1}(T_w u_{xx})\|_{L_x^2} \lesssim \|w\|_{L_x^\infty} \|u\|_{\dot{H}_x^{1+s}}.$$

For the balanced frequency case, we have

$$\begin{aligned}\|D_x^s \partial_x^{-1}\Pi(w, u_{xx})\|_{L_x^2} &\lesssim \|D_x^1 \partial_x^{-1}\Pi(w, u_{xx})\|_{L_x^{\frac{2}{3-2s}}} \lesssim \|\Pi(w, u_{xx})\|_{L_x^{\frac{2}{3-2s}}} \lesssim \|D_x^s w\|_{L_x^2} \|D_x^{2-s} u\|_{L_x^{\frac{1}{1-s}}} \\ &\lesssim \|D_x^s w\|_{L_x^2} \|D_x^{\frac{3}{2}} u\|_{L_x^2} \lesssim \|w\|_{\dot{H}_x^s} \|u\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}}.\end{aligned}$$

Lastly, for the commutator term, we have

$$\|[D_x^s, u] w_x\|_{L_x^2} \lesssim \|u_x\|_{L_x^\infty} \|w\|_{\dot{H}_x^s}.$$

By applying the energy estimate from Lemma 5.5, we get that for every $t \in [0, T]$,

$$\begin{aligned}\|w(t)\|_{\dot{H}_x^s} &\le e^{\frac{C}{2}\int_0^t \|u_x(\tau)\|_{L_x^\infty}\, d\tau} \left(\|w_0\|_{\dot{H}_x^s} + \int_0^t e^{-\int_0^\tau \|u_x(\eta)\|_{L_x^\infty} d\eta} \|u(\tau)\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}} \|w(\tau)\|_{L_x^\infty \cap \dot{H}_x^s}\, d\tau \right) \\ &\lesssim \|w_0\|_{\dot{H}_x^s} + T\|u\|_{L_t^\infty X^s} \|w\|_{L_x^\infty \cap \dot{H}_x^s}.\end{aligned}$$

Next, to obtain an L^∞ estimate, it suffices to consider the low frequencies since by Sobolev embedding and Bernstein's inequalities,

$$\|w_{>0}\|_{L_x^\infty} \lesssim \|w\|_{\dot{H}_x^s}.$$

For the first source term, we decompose into paraproducts as before to estimate (by using Bernstein's inequalities, Sobolev embedding, and Coifman-Meyer estimates)

$$\begin{aligned}\|P_{\le 0}\partial_x^{-1}(T_{u_{xx}} w)\|_{L_x^\infty} &\lesssim \|P_{\le 0}\partial_x^{-1}(T_{u_{xx}} w)\|_{L_x^{\frac{1}{1-s}}} \lesssim \|D_x^{-\frac{s}{2}} u_x\|_{L_x^{\frac{2}{1-s}}} \|D_x^{\frac{s}{2}} w\|_{L_x^{\frac{2}{1-s}}} \\ &\lesssim \|u_x\|_{L_x^2} \|w\|_{\dot{H}_x^s} \lesssim \|w\|_{\dot{H}_x^s} \|u\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}}, \\ \|P_{\le 0}\partial_x^{-1}(T_w u_{xx})\|_{L_x^\infty} &\lesssim \|\partial_x^{-1}(T_w u_{xx})\|_{H_x^s} \lesssim \|w\|_{L_x^\infty} \|u\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}},\end{aligned}$$

and

$$\|P_{\le 0}\partial_x^{-1}\Pi(u_{xx}, w)\|_{L_x^\infty} \lesssim \|P_{\le 0}\Pi(u_{xx}, w)\|_{L_x^1} \lesssim \|u_x\|_{\dot{H}_x^{1-s}} \|w\|_{\dot{H}_x^s} \lesssim \|u\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}} \|w\|_{\dot{H}_x^s}.$$

Thus,

$$\|P_{\le 0}\partial_x^{-1}(w u_{xx})\|_{L_x^\infty} \lesssim \|w\|_{L_x^\infty} \|u\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}} + \|u\|_{\dot{H}_x^1 \cap \dot{H}_x^{1+s}} \|w\|_{\dot{H}_x^s}.$$

From Lemma 5.8, with T' sufficiently small, we have

$$\|w_{\le 0}\|_{L_{t,x}^\infty} \lesssim \|(w_0)_{\le 0}\|_{L_x^\infty} + T'\|u\|_{L_t^\infty X^s} \|w\|_{L_t^\infty \dot{H}_x^s}.$$

Putting everything together, for $t \in [0, T']$ we get

$$\|w(t)\|_{L_x^\infty \cap \dot{H}_x^s} \lesssim \|w_0\|_{L_x^\infty \cap \dot{H}_x^s} + T'\|u\|_{L_t^\infty X^s} \|w\|_{L_x^\infty \cap \dot{H}_x^s}.$$

By further shrinking T' depending on $\|u\|_{L_t^\infty X^s}$, we obtain the desired estimate. $\square$

We now prove a result regarding differences of solutions, that is going to be used in order to justify uniqueness of $C_t^0 X^s$-solutions in the proof of Theorem 5.1.3.

Lemma 5.17. Let $T > 0$ and $I = [0, T]$. Let $u, v \in C_t^0 X^s(I \times \mathbb{R})$ solve (5.1.1) with $u_0 - v_0 \in L_x^\infty \cap \dot{H}_x^s$. Then $u - v \in L_t^\infty(L_x^\infty \cap \dot{H}_x^s)(I \times \mathbb{R})$, and for T sufficiently small depending on $\|(u, v)\|_{L_t^\infty X^s}$, we have

$$\|u - v\|_{L_t^\infty(L_x^\infty \cap \dot{H}_x^s)} \lesssim \|u_0 - v_0\|_{L_x^\infty \cap \dot{H}_x^s}.$$

Proof. Let $z = u - v$, which solves the equation

$$z_t + u z_x + v_x z + z_{xxx} = \frac{\partial_x^{-1}(z_x(u_x + v_x))}{2}. \tag{5.17.1}$$

We apply D_x^s and rearrange to consider the Cauchy problem

$$w_t + w_{xxx} = D_x^s\left(\frac{\partial_x^{-1}(z_x(u_x + v_x))}{2}\right) - D_x^s(uz)_x + D_x^s(z z_x) := H$$

with initial data $w(0) = D_x^s(u_0 - v_0) \in L_x^2(\mathbb{R})$. For the first term in H, we decompose into paraproducts and have the L^2 bounds

$$\begin{aligned}
\|D_x^s \partial_x^{-1}(T_{z_x}(u_x + v_x))\|_{L_x^2} &\lesssim \|z\|_{L_x^\infty} \|u + v\|_{\dot{H}_x^{1+s}} \\
\|D_x^s \partial_x^{-1}(T_{u_x + v_x} z_x)\|_{L_x^2} &\lesssim \|z\|_{\dot{H}_x^{1+s}} \|u + v\|_{L_x^\infty} \\
\|D_x^s \partial_x^{-1} \Pi(u_x + v_x, z_x)\|_{L_x^2} &\lesssim \|D_x \partial_x^{-1} \Pi(u_x + v_x, z_x)\|_{L_x^{\frac{2}{3-2s}}} \\
&\lesssim \|z\|_{\dot{H}_x^1} \|D_x(u + v)\|_{L_x^{\frac{1}{1-s}}} \lesssim \|z\|_{\dot{H}_x^1} \|D_x^{s+1/2}(u + v)\|_{L_x^2}.
\end{aligned}$$

The other terms are estimated directly using product estimates:

$$\begin{aligned}
\|D_x^s \partial_x(uz)\|_{L_x^2} &\lesssim \|u\|_{L_x^\infty} \|z\|_{\dot{H}_x^{1+s}} + \|z\|_{L_x^\infty} \|u\|_{\dot{H}_x^{1+s}} \\
\|D_x^s \partial_x(z^2)\|_{L_x^2} &\lesssim \|z\|_{L_x^\infty} \|z\|_{\dot{H}_x^{1+s}}.
\end{aligned}$$

Thus, $H \in L_t^\infty L_x^2([0, T] \times \mathbb{R})$. By applying Lemma 5.5, we infer that (5.17.1) has a unique solution in $L_t^\infty L_x^2([0, T] \times \mathbb{R})$. However, both w and $D_x^s z$ are solutions (in the sense of tempered distributions), hence $w = D_x^s z$, and $z = u - v \in L_t^\infty \dot{H}_x^s([0, T] \times \mathbb{R})$. It is also clear that $u - v \in L_{t,x}^\infty([0, T] \times \mathbb{R})$.

We now observe that z satisfies the linearized equation (5.15.1) with source,

$$z_t + \partial_x^{-1}(u_{xx} z) + u z_x + z_{xxx} = z z_x - \frac{\partial_x^{-1}(z_x^2)}{2} =: f.$$

After taking T small enough (depending on $\|u\|_{L_t^\infty X^s}$), we can apply Lemma 5.16, But first, we have to estimate $f \in L_t^1(L_x^\infty \cap \dot{H}_x^s)([0, T] \times \mathbb{R})$. For the first term of f,

$$\begin{aligned}
\|D_x^s \partial_x(z^2)\|_{L_x^2} &\lesssim \|z\|_{L_x^\infty}(\|u\|_{X^s} + \|v\|_{X^s}), \\
\|\partial_x(z^2)\|_{L_x^\infty} &\lesssim \|z\|_{L_x^\infty}(\|u\|_{X^s} + \|v\|_{X^s}).
\end{aligned}$$

For the second, by using Sobolev embedding and Bernstein's inequalities, we have

$$\begin{aligned}\|D_x^s\partial_x^{-1}(T_{z_x}z_x)\|_{L_x^2} &\lesssim \|z\|_{\dot{H}_x^s}\|z_x\|_{L_x^\infty} \lesssim \|z\|_{\dot{H}_x^s}(\|u\|_{X^s}+\|v\|_{X^s})\\ \|D_x^s\partial_x^{-1}\Pi(z_x,z_x)\|_{L_x^2} &\lesssim \|\Pi(z_x,z_x)\|_{L_x^{\frac{2}{3-2s}}} \lesssim \|D_x^s z\|_{L_x^2}\|D_x^{2-s}z\|_{L_x^{\frac{1}{1-s}}} \lesssim \|z\|_{\dot{H}_x^s}\|D_x^{3/2}z\|_{L_x^2}\\ &\lesssim \|z\|_{\dot{H}_x^s}(\|u\|_{X^s}+\|v\|_{X^s}).\end{aligned}$$

For the L_x^∞ estimate, it suffices to consider the low frequencies since by Sobolev embedding and Bernstein's inequalities,

$$\|\partial_x^{-1}(z_x^2)_{>0}\|_{L_x^\infty} \lesssim \|\partial_x^{-1}(z_x^2)_{>0}\|_{\dot{H}_x^s} \lesssim \|D_x^s\partial_x^{-1}(z_x^2)\|_{L_x^2} \lesssim \|z\|_{\dot{H}_x^s}(\|u\|_{X^s}+\|v\|_{X^s}).$$

We then have for the low frequencies

$$\begin{aligned}\|P_{\le 0}\partial_x^{-1}(\Pi(z_x,z_x))\|_{L_x^\infty} &\lesssim \|P_{\le 0}\Pi(z_x,z_x)\|_{L_x^1} \lesssim \|z\|_{\dot{H}_x^s}\|z\|_{\dot{H}_x^{2-s}} \lesssim \|z\|_{\dot{H}_x^s}(\|u\|_{X^s}+\|v\|_{X^s})\\ \|P_{\le 0}\partial_x^{-1}(T_{z_x}z_x)\|_{L_x^\infty} &\lesssim \|\partial_x^{-1}(T_{z_x}z_x)\|_{H_x^s} \lesssim \|z\|_{L_x^\infty}(\|u\|_{X^s}+\|v\|_{X^s}).\end{aligned}$$

Thus,

$$\|f\|_{L_t^1(L_x^\infty\cap\dot{H}_x^s)} \lesssim T\|z\|_{L_t^\infty(L_x^\infty\cap\dot{H}_x^s)}(\|u\|_{L_t^\infty X^s}+\|v\|_{L_t^\infty X^s}).$$

Thus, we get that

$$\begin{aligned}\|w\|_{L_t^\infty(L_x^\infty\cap\dot{H}_x^s)} &\lesssim_{\|u\|_{L_t^\infty X^s}} \|w_0\|_{L_x^\infty\cap\dot{H}_x^s} + \|f\|_{L_t^1(L_x^\infty\cap\dot{H}_x^s)}\\ &\lesssim_{\|u\|_{L_t^\infty X^s}} \|w_0\|_{L_x^\infty\cap\dot{H}_x^s} + T\|z\|_{L_t^\infty(L_x^\infty\cap\dot{H}_x^s)}(\|u\|_{L_t^\infty X^s}+\|v\|_{L_t^\infty X^s}).\end{aligned}$$

After further shrinking T (depending on $\|(u,v)\|_{L_t^\infty X^s}$), Lemma 5.16 implies the desired conclusion. □

5.18 Local well-posedness at low regularity

In this section, we prove Theorem 5.1.3. As we have already noticed at the end of Section 5.11, this will also imply Theorem 5.1.4.

Let $R>0$ be arbitrary. Given data u_0 satisfying $\|u_0\|_{X^s}<R$, we consider the corresponding regularized data

$$u_0^h = P_{<h}u_0.$$

Since $u_0^h \to u_0$ in X^s, we may assume that $\|u_0^h\|_{X^s}<R$ for all h.

We construct a uniform $\dot{H}_x^1\cap\dot{H}_x^{1+s}$ frequency envelope $\{c_k\}_{k\ge 0}$ for u_0 having the following properties:

a) Uniform bounds:

$$\|P_k(u_0^h)_x\|_{H_x^1\cap\dot{H}_x^{1+s}} \lesssim c_k$$

b) High frequency bounds:
$$\|u_0^h\|_{\dot{H}_x^1\cap\dot{H}_x^{2+s}} \lesssim 2^h c_h$$

c) Difference bounds:
$$\|u_0^{h+1} - u_0^h\|_{\dot{H}^s} \lesssim 2^{-h} c_h$$

d) Limit as $h \to \infty$:
$$D_x(u_0^h) \to D_x(u_0) \in H_x^s$$

By Theorem 5.1.2 and Lemma 5.7, u_0^h generate global smooth solutions u^h. Corollary 5.14 enables us to pick $T = T(R) > 0$ such that the hypotheses of Lemma 5.17 can be applied to any $C_t^0 X^s([0,T]\times\mathbb{R})$-solutions with initial data whose X^s-norm is smaller than R. Moreover, we also obtain uniform bounds for such solutions, including the family $(u^h)_{h\in\mathbb{Z}}$. We now get (from interpolating the higher energy estimates in Lemma 5.7) that

$$\|u^h\|_{C_t^0(\dot{H}_x^1\cap\dot{H}_x^{2+s})} \lesssim \|u_0^h\|_{\dot{H}_x^1\cap\dot{H}_x^{2+s}} \lesssim 2^h\|u_0^h\|_{\dot{H}_x^1\cap\dot{H}_x^{1+s}} \lesssim 2^h c_h,$$

and
$$\|u^{h+1} - u^h\|_{C_t^0\dot{H}_x^s} \lesssim 2^{-h} c_h$$

By interpolation, we infer that
$$\|u^{h+1} - u^h\|_{C_t^0\dot{H}_x^{1+s}} \lesssim c_h.$$

Thus, for $h \geq 0$,
$$\|u^{h+1} - u^h\|_{C_t^0(\dot{H}_x^1\cap\dot{H}_x^{1+s})} \lesssim c_h$$

As in [87], we get that
$$\|P_k u^h\|_{C_t^0(\dot{H}_x^1\cap\dot{H}_x^{1+s})} \lesssim c_k.$$

and that
$$\|u^{h+k} - u^h\|_{C_t^0(\dot{H}_x^1\cap\dot{H}_x^{1+s})} \lesssim c_{h\leq\cdot<h+k} = \left(\sum_{n=h}^{h+k-1} c_n^2\right)^{\frac{1}{2}}$$

for every $k \geq 1$. Thus, u^h converges to an element u belonging to $C_t^0(\dot{H}_x^1\cap\dot{H}_x^{1+s})([0,T]\times\mathbb{R})$. Moreover, we also obtain

$$\|u^h - u\|_{C_t^0(\dot{H}_x^1\cap\dot{H}_x^{1+s})} \lesssim c_{\geq h} = \left(\sum_{n=h}^{\infty} c_n^2\right)^{\frac{1}{2}}. \tag{5.18.1}$$

For pointwise convergence, we use Sobolev embedding for the high frequencies,

$$\|(u^{h+k})_{>0} - (u^h)_{>0}\|_{C_t^0 L_x^\infty} \lesssim \|u^{h+k} - u^h\|_{C_t^0\dot{H}_x^1}.$$

and the estimate (5.10.9) for the low frequencies (in which $E^{1/2}$ is bounded by R from above):

$$\|(u^{h+k})_{\leq 0} - (u^h)_{\leq 0}\|_{C_t^0 L_x^\infty} \lesssim \|u_0^{h+k} - u_0^h\|_{L_x^\infty} + TR\|u^{h+k} - u^h\|_{C_t^0 \dot{H}_x^1}.$$

We conclude that $u^h \to u \in C_t^0 X^s([0,T] \times \mathbb{R})$.

Lemma 5.17 also implies uniqueness for (5.1.1). For continuity with respect to the initial data, consider a sequence

$$u_{0j} \to u_0 \in X^s$$

and an associated sequence of $\dot{H}_x^1 \cap \dot{H}_x^{1+s}$-frequency envelopes $\{c_k^j\}_{k\geq 0}$, each satisfying the analogous properties enumerated above for c_k, and further such that $c^j \to c$ in $l^2(\mathbb{Z})$.

We may assume that $\|u_{0j}\|_{X^s} < R$ for every $j \geq 0$. As before, we get uniform bounds for $(u_j^h)_{(j,h)\in\mathbb{N}\times\mathbb{Z}}$, and we can interpolate to conclude

$$\|u_j^{h+1} - u_j^h\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} \lesssim c_h^j$$

and

$$\|P_k u_j^h\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} \lesssim c_k^j,$$

$$\|u_j^{h+k} - u_j^h\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} \lesssim c_{h\leq \cdot < h+k}^j = \left(\sum_{n=h}^{h+k-1} (c_n^j)^2\right)^{\frac{1}{2}},$$

$$\|u_j^h - u_j\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} \lesssim c_{\geq h}^j = \left(\sum_{n=h}^{\infty} (c_n^j)^2\right)^{\frac{1}{2}}.$$

Using the triangle inequality, we write

$$\begin{aligned}\|u_j - u\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} &\lesssim \|u^h - u\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} + \|u_j^h - u_j\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} + \|u_j^h - u^h\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})} \\ &\lesssim c_{\geq h} + c_{\geq h}^j + \|u_j^h - u^h\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})}\end{aligned}$$

For every fixed h, Theorem 5.1.2 tells us that $u_j^h \to u^h$ in X. This implies that $u_j \to u$ in $C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})([0,T] \times \mathbb{R})$. For pointwise estimates, by applying Sobolev embeddings and using Bernstein's inequalities, we get that

$$\|(u_j)_{>0} - u_{>0}\|_{C_t^0 L_x^\infty} \lesssim \|u_j - u\|_{C_t^0(\dot{H}_x^1 \cap \dot{H}_x^{1+s})}.$$

Besides this, (5.10.9) (in which $E^{1/2}$ is bounded by R from above) implies that

$$\|(u_j)_{\leq 0} - u_{\leq 0}\|_{C_t^0 L_x^\infty} \lesssim \|(u_j)_{\leq 0} - u_{\leq 0}\|_{C_t^0 L_x^\infty} + CTR\|u_j - u\|_{C_t^0 \dot{H}_x^1}.$$

Therefore, $u_j \to u$ in $C_t^0 X^s([0,T] \times \mathbb{R})$. This finishes the proof.

Bibliography

[1] A D Agaltsov, T Hohage, and R G Novikov. Global uniqueness in a passive inverse problem of helioseismology. *Inverse Problems*, 36(5):055004, apr 2020.

[2] Alexey D. Agaltsov, Thorsten Hohage, and Roman G. Novikov. Monochromatic identities for the green function and uniqueness results for passive imaging. *SIAM Journal on Applied Mathematics*, 78(5):2865–2890, 2018.

[3] Albert Ai and Ovidiu-Neculai Avadanei. Low regularity solutions for the surface quasi-geostrophic front equation. *arXiv e-prints*, page arXiv:2310.20143, October 2023.

[4] Albert Ai and Ovidiu-Neculai Avadanei. Low regularity well-posedness for the generalized surface quasi-geostrophic front equation. *arXiv e-prints, under revision at Pure and Applied Analysis*, page arXiv:2311.07551, November 2023.

[5] Albert Ai and Ovidiu-Neculai Avadanei. Well-posedness for the dispersive Hunter-Saxton equation. *Int. Math. Res. Not. IMRN*, (9):7883–7924, 2023.

[6] Albert Ai and Ovidiu-Neculai Avadanei. Well-posedness for the surface quasi-geostrophic front equation. *Nonlinearity*, 37(5):Paper No. 055022, 41, 2024.

[7] Albert Ai, Mihaela Ifrim, and Daniel Tataru. Two dimensional gravity waves at low regularity I: Energy estimates. *arXiv e-prints, to appear in Ann. Inst. H. Poincaré C Anal. Non Linéaire*, page arXiv:1910.05323, October 2019.

[8] Albert Ai, Mihaela Ifrim, and Daniel Tataru. The time-like minimal surface equation in Minkowski space: low regularity solutions. *Invent. Math.*, 235(3):745–891, 2024.

[9] T. Alazard, N. Burq, and C. Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[10] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198(1):71–163, 2014.

[11] T. Alazard, N. Burq, and C. Zuily. On the Cauchy problem for gravity water waves. *Invent. Math.*, 198(1):71–163, 2014.

[12] Thomas Alazard and Jean-Marc Delort. Global solutions and asymptotic behavior for two dimensional gravity water waves. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(5):1149–1238, 2015.

[13] Lars Andersson and Huali Zhang. Well-posedness for rough solutions of the 3D compressible Euler equations. *arXiv e-prints*, page arXiv:2208.10132, August 2022.

[14] Ovidiu-Neculai Avadanei. Counterexamples to Strichartz estimates and gallery waves for the irrotational compressible Euler equation in a vacuum setting. *arXiv e-prints*, page arXiv:2504.17932, April 2025.

[15] Hajer Bahouri and Jean-Yves Chemin. équations d'ondes quasilinéaires et effet dispersif. *Internat. Math. Res. Notices*, (21):1141–1178, 1999.

[16] Hajer Bahouri and Jean-Yves Chemin. équations d'ondes quasilinéaires et estimations de Strichartz. *Amer. J. Math.*, 121(6):1337–1377, 1999.

[17] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[18] Richard Beals, David H. Sattinger, and Jacek Szmigielski. Inverse scattering solutions of the Hunter-Saxton equation. *Appl. Anal.*, 78(3-4):255–269, 2001.

[19] Bensalah, Antoine, Joly, Patrick, and Mercier, Jean-Francois. Mathematical analysis of goldstein's model for time-harmonic acoustics in flows. *ESAIM: M2AN*, 56(2):451–483, 2022.

[20] Massimiliano Berti, Scipio Cuccagna, Francisco Gancedo, and Stefano Scrobogna. Paralinearization and extended lifespan for solutions of the α-SQG sharp front equation. *arXiv e-prints*, page arXiv:2310.15963, October 2023.

[21] A. L. Bertozzi and P. Constantin. Global regularity for vortex patches. *Comm. Math. Phys.*, 152(1):19–28, 1993.

[22] Andrea Louise Bertozzi. *Existence, uniqueness, and a characterization of solutions to the contour dynamics equation*. ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–Princeton University.

[23] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. Strichartz estimates for the wave equation on manifolds with boundary. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 26(5):1817–1829, 2009.

[24] Alberto Bressan and Adrian Constantin. Global solutions of the Hunter-Saxton equation. *SIAM J. Math. Anal.*, 37(3):996–1026, 2005.

[25] Alberto Bressan, Helge Holden, and Xavier Raynaud. Lipschitz metric for the Hunter-Saxton equation. *J. Math. Pures Appl. (9)*, 94(1):68–92, 2010.

[26] Nicolas Burq, Gilles Lebeau, and Fabrice Planchon. Global existence for energy critical waves in 3-D domains. *J. Amer. Math. Soc.*, 21(3):831–845, 2008.

[27] Jean-Yves Chemin. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. École Norm. Sup. (4)*, 26(4):517–542, 1993.

[28] Gui-Qiang Chen. Remarks on R. J. DiPerna's paper: "Convergence of the viscosity method for isentropic gas dynamics" [Comm. Math. Phys. **9**1 (1983), no. 1, 1–30; MR0719807 (85i:35118)]. *Proc. Amer. Math. Soc.*, 125(10):2981–2986, 1997.

[29] Alexandre J. Chorin and Jerrold E. Marsden. *A mathematical introduction to fluid mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition, 1993.

[30] Michael Christ, James Colliander, and Terrence Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[31] J. Christensen-Dalsgaard, W. Däppen, S. V. Ajukov, E. R. Anderson, H. M. Antia, S. Basu, V. A. Baturin, G. Berthomieu, B. Chaboyer, S. M. Chitre, A. N. Cox, P. Demarque, J. Donatowicz, W. A. Dziembowski, M. Gabriel, D. O. Gough, D. B. Guenther, J. A. Guzik, J. W. Harvey, F. Hill, G. Houdek, C. A. Iglesias, A. G. Kosovichev, J. W. Leibacher, P. Morel, C. R. Proffitt, J. Provost, J. Reiter, E. J. Rhodes, F. J. Rogers, I. W. Roxburgh, M. J. Thompson, and R. K. Ulrich. The current state of solar modeling. *Science*, 272(5266):1286–1292, 1996.

[32] Jorgen Christensen-Dalsgaard, Douglas Gough, and Juri Toomre. Seismology of the sun. *Science*, 229(4717):923–931, 1985.

[33] Demetrios Christodoulou and Shuang Miao. *Compressible flow and Euler's equations*, volume 9 of *Surveys of Modern Mathematics*. International Press, Somerville, MA; Higher Education Press, Beijing, 2014.

[34] R. R. Coifman and Yves Meyer. On commutators of singular integrals and bilinear singular integrals. *Trans. Amer. Math. Soc.*, 212:315–331, 1975.

[35] Adrian Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier (Grenoble)*, 50(2):321–362, 2000.

[36] Adrian Constantin and Joachim Escher. Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(2):303–328, 1998.

[37] Adrian Constantin and Joachim Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.*, 181(2):229–243, 1998.

[38] Adrian Constantin and Joachim Escher. Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. *Comm. Pure Appl. Math.*, 51(5):475–504, 1998.

[39] Diego Córdoba, Javier Gómez-Serrano, and Alexandru D. Ionescu. Global solutions for the generalized SQG patch equation. *Arch. Ration. Mech. Anal.*, 233(3):1211–1251, 2019.

[40] Daniel Coutand, Hans Lindblad, and Steve Shkoller. A priori estimates for the free-boundary 3D compressible Euler equations in physical vacuum. *Comm. Math. Phys.*, 296(2):559–587, 2010.

[41] Daniel Coutand and Steve Shkoller. Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible Euler equations in physical vacuum. *Arch. Ration. Mech. Anal.*, 206(2):515–616, 2012.

[42] Daniel Coutand and Steve Shkoller. Regularity of the velocity field for Euler vortex patch evolution. *Trans. Amer. Math. Soc.*, 370(5):3689–3720, 2018.

[43] Raphaël Danchin. A note on well-posedness for Camassa-Holm equation. *J. Differential Equations*, 192(2):429–444, 2003.

[44] Thibault de Poyferré. A priori estimates for water waves with emerging bottom. *Arch. Ration. Mech. Anal.*, 232(2):763–812, 2019.

[45] A.S.Bonnet-Ben Dhia, J.F. Mercier, F. Millot, S. Pernet, and E. Peynaud. Time-harmonic acoustic scattering in a complex flow: A full coupling between acoustics and hydrodynamics. *Communications in Computational Physics*, 11(2):555–572, 2012.

[46] Ronald J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1):1–30, 1983.

[47] Marcelo M. Disconzi, Mihaela Ifrim, and Daniel Tataru. The relativistic Euler equations with a physical vacuum boundary: Hadamard local well-posedness, rough solutions, and continuation criterion. *Archive for Rational Mechanics and Analysis*, 245(1):127–182, July 2022.

[48] Marcelo M. Disconzi, Chenyun Luo, Giusy Mazzone, and Jared Speck. Rough sound waves in 3D compressible Euler flow with vorticity. *Selecta Math. (N.S.)*, 28(2):Paper No. 41, 153, 2022.

[49] T. L. Duvall, Jr., S. M. Jefferies, J. W. Harvey, and M. A. Pomerantz. Time-distance helioseismology. *Nature*, 362(6419):430–432, April 1993.

[50] David G. Ebin. The equations of motion of a perfect fluid with free boundary are not well posed. *Comm. Partial Differential Equations*, 12(10):1175–1201, 1987.

[51] David G. Ebin and Jerrold E. Marsden. Groups of diffeomorphisms and the solution of the classical Euler equations for a perfect fluid. *Bull. Amer. Math. Soc.*, 75:962–967, 1969.

[52] Mingwen Fei. The integrability of dispersive Hunter-Saxton equation. *J. Partial Differ. Equ.*, 25(4):330–334, 2012.

[53] Henri Galbrun. Propagation d'une onde sonore dans l'atmosphère et théorie des zones de silence. 1931.

[54] Francisco Gancedo. Existence for the α-patch model and the QG sharp front in Sobolev spaces. *Adv. Math.*, 217(6):2569–2598, 2008.

[55] Francisco Gancedo, Huy Q. Nguyen, and Neel Patel. Well-posedness for SQG sharp fronts with unbounded curvature. *arXiv e-prints*, page arXiv:2105.10982, May 2021.

[56] Francisco Gancedo and Neel Patel. On the local existence and blow-up for generalized SQG patches. *Ann. PDE*, 7(1):Paper No. 4, 63, 2021.

[57] Francisco Gancedo and Robert M. Strain. Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem. *Proc. Natl. Acad. Sci. USA*, 111(2):635–639, 2014.

[58] J. Ginibre and G. Velo. The global Cauchy problem for the non linear Schrödinger equation revisited. *Annales de l'I.H.P. Analyse non linéaire*, 2(4):309–327, 1985.

[59] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.*, 133(1):50–68, 1995.

[60] Laurent Gizon, Aaron C. Birch, and Henk C. Spruit. Local Helioseismology: Three-Dimensional Imaging of the Solar Interior. *ARAA*, 48:289–338, September 2010.

[61] Laurent Gizon, Robert H. Cameron, Majid Pourabdian, Zhi-Chao Liang, Damien Fournier, Aaron C. Birch, and Chris S. Hanson. Meridional flow in the sun's convection zone is a single cell in each hemisphere. *Science*, 368(6498):1469–1472, 2020.

[62] Gizon, Laurent, Barucq, Hélène, Duruflé, Marc, Hanson, Chris S., Leguèbe, Michael, Birch, Aaron C., Chabassier, Juliette, Fournier, Damien, Hohage, Thorsten, and Papini, Emanuele. Computational helioseismology in the frequency domain: acoustic waves in axisymmetric solar models with flows. *A&A*, 600:A35, 2017.

[63] Zihua Guo, Xingxing Liu, Luc Molinet, and Zhaoyang Yin. Ill-posedness of the Camassa-Holm and related equations in the critical space. *J. Differential Equations*, 266(2-3):1698–1707, 2019.

[64] Martin Halla. Convergence analysis of nonconform $h(\text{div})$-finite elements for the damped time-harmonic galbrun's equation. 06 2023.

[65] Martin Halla and Thorsten Hohage. On the well-posedness of the damped time-harmonic galbrun equation and the equations of stellar oscillations. *SIAM Journal on Mathematical Analysis*, 53(4):4068–4095, 2021.

[66] Martin Halla, Christoph Lehrenfeld, and Paul Stocker. A new t-compatibility condition and its application to the discretization of the damped time-harmonic galbrun's equation. 09 2022.

[67] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[68] Thomas J. R. Hughes, Tosio Kato, and Jerrold E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.*, 63(3):273–294 (1977), 1976.

[69] Thomas J. R. Hughes, Tosio Kato, and Jerrold E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.*, 63(3):273–294 (1977), 1976.

[70] John K. Hunter, Mihaela Ifrim, and Daniel Tataru. Two dimensional water waves in holomorphic coordinates. *Comm. Math. Phys.*, 346(2):483–552, 2016.

[71] John K. Hunter, Mihaela Ifrim, Daniel Tataru, and Tak Kwong Wong. Long time solutions for a Burgers-Hilbert equation via a modified energy method. *Proc. Amer. Math. Soc.*, 143(8):3407–3412, 2015.

[72] John K. Hunter, Mihaela Ifrim, Daniel Tataru, and Tak Kwong Wong. Long time solutions for a Burgers-Hilbert equation via a modified energy method. *Proc. Amer. Math. Soc.*, 143(8):3407–3412, 2015.

[73] John K. Hunter and Ralph Saxton. Dynamics of director fields. *SIAM J. Appl. Math.*, 51(6):1498–1521, 1991.

[74] John K. Hunter and Jingyang Shu. Regularized and approximate equations for sharp fronts in the surface quasi-geostrophic equation and its generalizations. *Nonlinearity*, 31(6):2480–2517, 2018.

[75] John K. Hunter, Jingyang Shu, and Qingtian Zhang. Local well-posedness of an approximate equation for SQG fronts. *J. Math. Fluid Mech.*, 20(4):1967–1984, 2018.

[76] John K. Hunter, Jingyang Shu, and Qingtian Zhang. Contour dynamics for surface quasi-geostrophic fronts. *Nonlinearity*, 33(9):4699–4714, 2020.

[77] John K. Hunter, Jingyang Shu, and Qingtian Zhang. Global solutions of a surface quasigeostrophic front equation. *Pure Appl. Anal.*, 3(3):403–472, 2021.

[78] John K. Hunter, Jingyang Shu, and Qingtian Zhang. Global solutions for a family of GSQG front equations. *Discrete Contin. Dyn. Syst.*, 44(9):2598–2640, 2024.

[79] John K. Hunter and Yu Xi Zheng. On a completely integrable nonlinear hyperbolic variational equation. *Phys. D*, 79(2-4):361–386, 1994.

[80] John K. Hunter and Yu Xi Zheng. On a nonlinear hyperbolic variational equation. II. The zero-viscosity and dispersion limits. *Arch. Rational Mech. Anal.*, 129(4):355–383, 1995.

[81] Linus Hägg and Martin Berggren. On the well-posedness of galbrun's equation. *Journal de Mathématiques Pures et Appliquées*, 150:112–133, 2021.

[82] Mihaela Ifrim, Herbert Koch, and Daniel Tataru. Dispersive decay of small data solutions for the KdV equation. *arXiv e-prints*, page arXiv:1901.05934, January 2019.

[83] Mihaela Ifrim, Ben Pineau, Daniel Tataru, and Mitchell A. Taylor. Sharp Hadamard local well-posedness, enhanced uniqueness and pointwise continuation criterion for the incompressible free boundary Euler equations. *arXiv e-prints, to appear in Annals of PDE*, page arXiv:2309.05625, September 2023.

[84] Mihaela Ifrim and Daniel Tataru. Global bounds for the cubic nonlinear Schrödinger equation (NLS) in one space dimension. *Nonlinearity*, 28(8):2661–2675, 2015.

[85] Mihaela Ifrim and Daniel Tataru. Well-posedness and dispersive decay of small data solutions for the Benjamin-Ono equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 52(2):297–335, 2019.

[86] Mihaela Ifrim and Daniel Tataru. Testing by wave packets and modified scattering in nonlinear dispersive pde's. *arXiv e-prints*, page arXiv:2204.13285, April 2022.

[87] Mihaela Ifrim and Daniel Tataru. Local well-posedness for quasi-linear problems: a primer. *Bull. Amer. Math. Soc. (N.S.)*, 60(2):167–194, 2023.

[88] Mihaela Ifrim and Daniel Tataru. The compressible Euler equations in a physical vacuum: A comprehensive Eulerian approach. *Annales de L'Institut Henri Poincare Section (C) Non Linear Analysis*, 41(2):405–495, August 2023.

[89] Alexandru D. Ionescu and Carlos E. Kenig. Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. *J. Amer. Math. Soc.*, 20(3):753–798, 2007.

[90] Oana Ivanovici. Counterexamples to Strichartz estimates for the wave equation in domains. *Math. Ann.*, 347(3):627–673, 2010.

[91] Oana Ivanovici. Dispersive estimates for the wave and the Klein-Gordon equations in large time inside the Friedlander domain. *Discrete Contin. Dyn. Syst.*, 41(12):5707–5742, 2021.

[92] Oana Ivanovici, Richard Lascar, Gilles Lebeau, and Fabrice Planchon. Dispersion for the wave equation inside strictly convex domains II: The general case. *Ann. PDE*, 9(2):Paper No. 14, 117, 2023.

[93] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case. *Ann. of Math. (2)*, 180(1):323–380, 2014.

[94] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Estimations de Strichartz pour l'équation des ondes dans un domaine strictement convexe. In *PDE's, dispersion, scattering theory and control theory*, volume 30 of *Sémin. Congr.*, pages 69–79. Soc. Math. France, Paris, 2017.

[95] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. New counterexamples to Strichartz estimates for the wave equation on a 2D model convex domain. *Journal de l'École polytechnique — Mathématiques*, 8:1133–1157, 2021.

[96] Juhi Jang and Nader Masmoudi. Well-posedness for compressible Euler equations with physical vacuum singularity. *Comm. Pure Appl. Math.*, 62(10):1327–1385, 2009.

[97] Juhi Jang and Nader Masmoudi. Well-posedness of compressible Euler equations in a physical vacuum. *Comm. Pure Appl. Math.*, 68(1):61–111, 2015.

[98] V. I. Judovi**v**c. Non-stationary flows of an ideal incompressible fluid. ***v**Z. Vy**v**cisl. Mat i Mat. Fiz.*, 3:1032–1066, 1963.

[99] Christensen-Dalsgaard Jørgen. Lecture notes on stellar oscillations.

[100] L. V. Kapitanskii. Estimates for norms in Besov and Lizorkin-Triebel spaces for solutions of second-order linear hyperbolic equations. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 171:106–162, 185–186, 1989.

[101] L. V. Kapitanski**u** i. Some generalizations of the Strichartz-Brenner inequality. *Algebra i Analiz*, 1(3):127–159, 1989.

[102] Tosio Kato and Gustavo Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[103] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[104] Alexander Kiselev and Xiaoyutao Luo. The α-sqg patch problem is illposed in $c^{2,\beta}$ and $w^{2,p}$, 2023.

[105] Alexander Kiselev and Xiaoyutao Luo. On nonexistence of splash singularities for the α-SQG patches. *J. Nonlinear Sci.*, 33(2):Paper No. 37, 16, 2023.

[106] Herbert Koch, Hart F. Smith, and Daniel Tataru. Subcritical L^p bounds on spectral clusters for Lipschitz metrics. *Math. Res. Lett.*, 15(5):993–1002, 2008.

[107] Hans Lindblad. Well posedness for the motion of a compressible liquid with free surface boundary. *Comm. Math. Phys.*, 260(2):319–392, 2005.

[108] Hans Lindblad and Chenyun Luo. A priori estimates for the compressible Euler equations for a liquid with free surface boundary and the incompressible limit. *Comm. Pure Appl. Math.*, 71(7):1273–1333, 2018.

[109] Hans Lindblad and Christopher D. Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *J. Funct. Anal.*, 130(2):357–426, 1995.

[110] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[111] Tai-Ping Liu and Tong Yang. Compressible Euler equations with vacuum. *J. Differential Equations*, 140(2):223–237, 1997.

[112] D. Lynden-Bell and J. P. Ostriker. On the stability of differentially rotating bodies. *Monthly Notices of the Royal Astronomical Society*, 136(3):293–310, 07 1967.

[113] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[114] Tetu Makino, Seiji Ukai, and Shuichi Kawashima. Sur la solution à support compact de l'équations d'Euler compressible. *Japan J. Appl. Math.*, 3(2):249–257, 1986.

[115] Jerrold E. Marsden, Tudor Ratiu, and Alan Weinstein. Reduction and Hamiltonian structures on duals of semidirect product Lie algebras. In *Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983)*, volume 28 of *Contemp. Math.*, pages 55–100. Amer. Math. Soc., Providence, RI, 1984.

[116] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.

[117] Björn Müller, Thorsten Hohage, Damien Fournier, and Laurent Gizon. Quantitative passive imaging by iterative holography: the example of helioseismic holography. *Inverse Problems*, 40(4):045016, mar 2024.

[118] Peter J. Olver and Philip Rosenau. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. *Phys. Rev. E (3)*, 53(2):1900–1906, 1996.

[119] Razvan-Octavian Radu. Existence and regularity for vortex patch solutions of the 2D Euler equations. *Indiana Univ. Math. J.*, 71(5):2195–2230, 2022.

[120] Serge G. Resnick. *Dynamical problems in non-linear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–The University of Chicago.

[121] José Luis Rodrigo. On the evolution of sharp fronts for the quasi-geostrophic equation. *Comm. Pure Appl. Math.*, 58(6):821–866, 2005.

[122] Philippe Serfati. Une preuve directe d'existence globale des vortex patches 2d. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(6):515–518, 1994.

[123] Jalal Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. *Comm. Pure Appl. Math.*, 38(5):685–696, 1985.

[124] Jalal Shatah and Chongchun Zeng. Geometry and a priori estimates for free boundary problems of the Euler equation. *Comm. Pure Appl. Math.*, 61(5):698–744, 2008.

[125] Jalal Shatah and Chongchun Zeng. A priori estimates for fluid interface problems. *Comm. Pure Appl. Math.*, 61(6):848–876, 2008.

[126] Jalal Shatah and Chongchun Zeng. Local well-posedness for fluid interface problems. *Arch. Ration. Mech. Anal.*, 199(2):653–705, 2011.

[127] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[128] Hart F. Smith and Christopher D. Sogge. On the critical semilinear wave equation outside convex obstacles. *J. Amer. Math. Soc.*, 8(4):879–916, 1995.

[129] Hart F. Smith and Christopher D. Sogge. On the L^p norm of spectral clusters for compact manifolds with boundary. *Acta Math.*, 198(1):107–153, 2007.

[130] Hart F. Smith and Daniel Tataru. Sharp counterexamples for Strichartz estimates for low regularity metrics. *Math. Res. Lett.*, 9(2-3):199–204, 2002.

[131] Hart F. Smith and Daniel Tataru. Sharp local well-posedness results for the nonlinear wave equation. *Ann. of Math. (2)*, 162(1):291–366, 2005.

[132] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[133] Terence Tao. Lecture notes 6 for 247b.

[134] Terence Tao. Lecture notes 8 for 247b.

[135] Terence Tao. Global regularity of wave maps. I. Small critical Sobolev norm in high dimension. *Internat. Math. Res. Notices*, (6):299–328, 2001.

[136] Terence Tao. Global regularity of wave maps. II. Small energy in two dimensions. *Comm. Math. Phys.*, 224(2):443–544, 2001.

[137] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[138] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. *Amer. J. Math.*, 122(2):349–376, 2000.

[139] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. II. *Amer. J. Math.*, 123(3):385–423, 2001.

[140] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442, 2002.

[141] Michael E. Taylor. *Pseudodifferential operators and nonlinear PDE*, volume 100 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1991.

[142] Chao Wang, Zhifei Zhang, Weiren Zhao, and Yunrui Zheng. Local well-posedness and break-down criterion of the incompressible Euler equations with free boundary. *Mem. Amer. Math. Soc.*, 270(1318):v + 119, 2021.

[143] Qian Wang. Rough solutions of the 3-D compressible Euler equations. *Ann. of Math. (2)*, 195(2):509–654, 2022.

[144] Weikui Ye and Zhaoyang Yin. Global existence and blow-up phenomena for the Hunter-Saxton equation on the line. *J. Math. Fluid Mech.*, 24(1):Paper No. 25, 14, 2022.

[145] Zhaoyang Yin. On the structure of solutions to the periodic Hunter-Saxton equation. *SIAM J. Math. Anal.*, 36(1):272–283, 2004.

[146] Huali Zhang. Low regularity solutions of two-dimensional compressible Euler equations with dynamic vorticity. *arXiv e-prints*, page arXiv:2012.01060, December 2020.

[147] Huali Zhang and Lars Andersson. On the rough solutions of 3D compressible Euler equations: an alternative proof. *arXiv e-prints*, page arXiv:2104.12299, April 2021.

[148] Andrej Zlatos. Local regularity and finite time singularity for the generalized SQG equation on the half-plane, May 2023.